

Pricing Models with Jumps in Credit Risk

by

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Abstract

Louis Bachelier (1900) suggested to model stock prices in terms of a Brownian motion with drift, which led to the Black-Scholes model where the log-prices follow a geometric Brownian motion. However, the price trajectories of financial derivatives such as stocks and bonds move in a discontinuous fashion: From one day to the next, the price will either stay the same, or move up or down by jumps. Brownian motion is a continuous stochastic process and cannot capture this property. In reality, prices may admit large abrupt moves and modeling in terms of a continuous process may result in a significant underestimation of risk. This is the most important argument for modeling derivative prices with jumps. Additionally, the typical empirical distribution of the log-returns of a stock has heavy tails (quite on the contrary to a Gaussian variable like Brownian motion), indicating that the probability of a large move cannot be ignored.

A problem with the classical firm value model of Merton (1974) arises from modeling the firm value in terms of a diffusion. The resulting term structure of the credit spreads slopes upwards from zero, even for financially stable firms, implying that their default risks are increasing with time. In reality credit spread curves can also slope downwards or be flat. Another issue is the expectancy of a default: With diffusion models, one has an increasing sequence of stopping times converging towards the default time. A firm can therefore never default unexpectedly with this approach. It is not possible for neither structural nor intensity based models based on diffusions to model both expected and unexpected defaults. The incorporation of jump-diffusions has been shown to generate the correct shapes of the yield spread curves and match the sizes of the credit spreads of corporate bonds. Furthermore, the possibility of an unexpected default of the firm is also taken care of by the jumps in the credit risk.

This thesis will be organized as follows: First, an introduction to the most basic concepts in stochastic analysis is given. The results are then utilized in the following chapters about modeling credit risk, where the theory of pricing and hedging of certain credit derivatives is presented. The need of including Lévy processes will become evident, and an introduction is given. The Vasicek intensity model (for both diffusions and jump processes) is calibrated to market data in order to price both default-free and defaultable bonds. Finally, an extension of the Vasicek model to a regime-switched version is discussed (more specifically in the setting of bond pricing) and calibrated to market data.

Remark: Sections marked with ♠ will denote results not found in any of the relevant literature, it thus marks my attempts to obtain new results.

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Chapter 1

Stochastic Analysis

The main idea of this chapter is to briefly define and explain some basic concepts of stochastic analysis, relating them to applications in finance. It will serve as a toolbox for the theory that will be discussed throughout this thesis. Most of the material is borrowed from [13].

1.1 Brownian motion

Standard Brownian motion is one of the simplest continuous-time stochastic processes. It has been widely applied to model random behaviour over time, as for example the evolution of stock prices or interest rates. Its definition is as follows:

Definition 1.1.1 (Brownian motion). *A stochastic process $\{W_t\}_{t \geq 0}$ is called a standard Brownian motion if it satisfies the following:*

1. $W_0 = 0$,
2. *For the time points $0 \leq s < t < u \leq v$, the increments $W_t - W_s$ and $W_v - W_u$ are independent random variables,*
3. *For every $h \geq 0$, the increment $W_{t+h} - W_t$ follows the Gaussian distribution with expectation 0 and variance h .*

In other words, Brownian motion starts at the origin, and moves in terms of independent and stationary increments following the normal distribution. Brownian motion is continuous almost everywhere, but nowhere differentiable. In the Black-Scholes model, (which consists of one risk-free asset and at least one risky asset) stock prices S_t are modeled in

terms of geometric Brownian motion, i.e.

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad (1.1)$$

where μ and σ describe the drift and volatility, respectively.

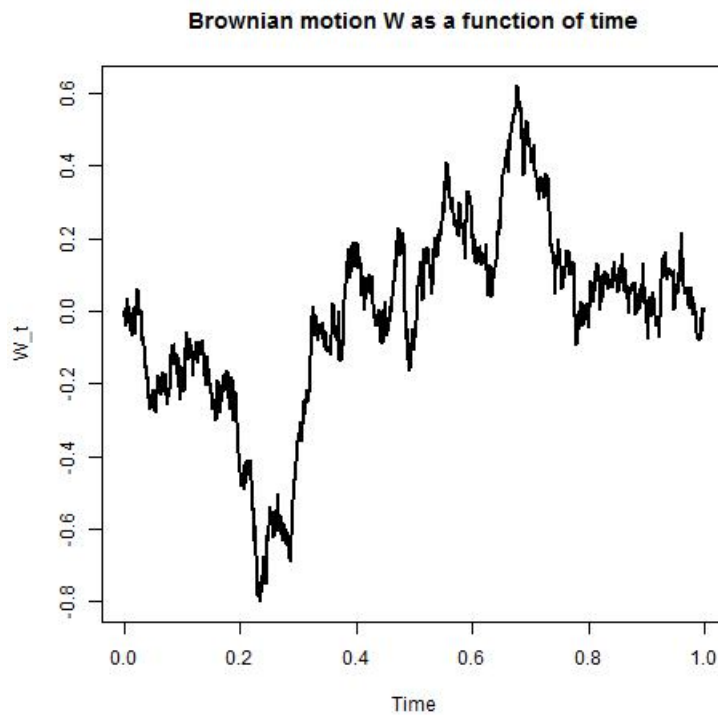


Figure 1.1: *A sample path of Brownian motion.*

1.2 The Itô Integral and Martingales

As Brownian motion is nowhere differentiable and of infinite variation, the methods of calculus no longer apply. More generally, the integral is taken with respect to a semimartingale, where the integrand is required to be locally square integrable wrt. the filtration generated by the semimartingale. In the case of Brownian motion, we have the following definition of the Itô integral:

Definition 1.2.1 (Itô integral). *Let $f(t, \omega)$ be a stochastic process with finite second moment, adapted to the filtration generated by W_t . The Itô integral $I[f]$ over the interval $[0, T]$*

is defined by

$$I[f] = \int_0^T f(t, \omega) dW_t := \lim_{n \rightarrow \infty} \sum_{[t_{i-1}, t_i] \in p_n} f(t_i, \omega) (W_{t_i} - W_{t_{i-1}}), \quad (1.2)$$

where p is the partition of $[0, T]$ with mesh going to zero as $n \rightarrow \infty$.

In the case of (1.1), the stochastic integral can be interpreted as the payoff from a trading strategy holding the amount $f(t, \omega)$ at time t of the stock. The left end points are used to evaluate the function: An investor thus first makes a decision, then thereafter observes the changes in the stock price. He cannot look into the future and ensure a profit.

Definition 1.2.2 (Itô process). *Let $a(t, \omega)$ and $b(t, \omega)$ be predictable stochastic processes on $(\Omega, \mathcal{F}_t, \mathbb{P})$ satisfying*

$$\mathbb{P}\left[\int_0^t (|a(s, \omega)| + b^2(s, \omega)) ds < \infty, \quad \forall t \geq 0\right] = 1. \quad (1.3)$$

Any stochastic process X_t on $(\Omega, \mathcal{F}_t, \mathbb{P})$ given by the representation

$$X_t = X_0 + \int_0^t a(s, \omega) ds + \int_0^t b(s, \omega) dW_s, \quad (1.4)$$

where W_t is 1D Brownian motion, is called an Itô process.

From now on,

$$dX_t = a(t, \omega) dt + b(t, \omega) dW_t \quad (1.5)$$

will be used as a short-hand notation for (1.4).

The *Itô formula* describes how to calculate the differential of a time-dependent function of an Itô process (which again is another Itô process by the following):

Theorem 1.2.1 (The 1D Itô formula [13]). *Let X_t be an Itô process given by (1.4). Let $g(t, x) \in C^2([0, \infty) \times \mathbb{R})$ (i.e. g is twice continuously differentiable on $[0, \infty) \times \mathbb{R}$). Then $Y_t = g(t, X_t)$ is again an Itô process, and*

$$dY_t = \frac{\partial g}{\partial t}(t, X_t) dt + \frac{\partial g}{\partial x}(t, X_t) dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X_t) \cdot (dX_t)^2, \quad (1.6)$$

where $(dX_t)^2 = (dX_t) \cdot (dX_t)$ is computed according to the rules

$$dt \cdot dt = dt \cdot dW_t = dW_t \cdot dt = 0, \quad , \quad dW_t \cdot dW_t = dt. \quad (1.7)$$

We now have established a way of calculating Itô integrals:

Example 1.2.1. Consider $I = \int_0^t W_s dW_s$. With $g(t, x) = \frac{1}{2}x^2$ and $X_t = W_t$ in Theorem 1.2.1:

$$d(g(t, W_t)) = d\left(\frac{1}{2}W_t^2\right) = 0 \cdot dt + W_t dW_t + \frac{1}{2}(dW_t)^2 = W_t dW_t + \frac{1}{2}dt$$

\Leftrightarrow

$$\frac{1}{2}W_t^2 = \int_0^t W_s dW_s + \frac{1}{2}t,$$

hence

$$\int_0^t W_s dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t.$$

Theorem (1.2.1) can be extended to hold for n-dimensional Brownian motion, see e.g. [13]. An important class of stochastic processes in finance are martingales, given by the next definition:

Definition 1.2.3 (Martingale). A stochastic process M_t on $(\Omega, \mathcal{F}_t, \mathbb{P})$ is called a martingale wrt. an underlying filtration $\mathcal{G}_t \subset \mathcal{F}$ (and \mathbb{P}) if it satisfies the following:

1. M_t is \mathcal{G}_t -measurable $\forall t \geq 0$.
2. $\mathbb{E}_{\mathbb{P}}[|M_t|] < \infty, \forall t \geq 0$.
3. $\mathbb{E}_{\mathbb{P}}[M_s | \mathcal{G}_t] = M_t, \forall s \geq t \geq 0$.

A martingale is thus a measurable stochastic process with finite first moment, whose expected value is equal to its last known value. It is easy to show that e.g. Brownian motion W_t is a martingale by checking the properties of Definition 1.2.3:

1. W_t generates the filtration \mathcal{F}_t by definition, where $\mathcal{G}_t \subset \mathcal{F}, \forall t \geq 0$.
2. By the Cauchy-Schwarz inequality, $\mathbb{E}_{\mathbb{P}}[|W_t|] \leq \sqrt{\mathbb{E}_{\mathbb{P}}[1^2] \mathbb{E}_{\mathbb{P}}[W_t^2]} = \sqrt{t} < \infty, \forall t \geq 0$.
3. $\mathbb{E}_{\mathbb{P}}[W_s | \mathcal{G}_t] = \mathbb{E}_{\mathbb{P}}[W_s - W_t + W_t | \mathcal{G}_t] = \mathbb{E}_{\mathbb{P}}[W_s - W_t | \mathcal{G}_t] + \mathbb{E}_{\mathbb{P}}[W_t | \mathcal{G}_t] = W_t$, where the last equality follows from property 1 and independence between the increment $W_s - W_t$ and the filtration \mathcal{G}_t . \square

Conversely, it can be shown that any Itô integral is a martingale, and that M_t is a martingale if and only if $E[M_t] = E[M_0]$. The following relation allows us to compute the variance of Itô integrals:

Proposition 1.2.1 (The Itô isometry). *Let $f(t, \omega)$ be a stochastic process with finite second moment, adapted to the filtration generated by W_t . Then,*

$$\mathbb{E}_{\mathbb{P}}[(\int_0^T f(t, \omega) dW_t)^2] = \mathbb{E}_{\mathbb{P}}[\int_0^T |f^2(t, \omega)| dt]. \quad (1.8)$$

Itô integrals follow the Gaussian distribution with expectation zero and variance as given by (1.8). Conversely, every martingale under certain integrability conditions admits a representation in terms of an Itô integral:

Theorem 1.2.2 (Martingale representation theorem). *Let M_t be a \mathcal{G}_t -martingale under the probability measure \mathbb{P} and assume that $\mathbb{E}_{\mathbb{P}}[M_t^2] < \infty$ for all $t \geq 0$. Then there exists a unique, predictable and \mathcal{G}_t -adapted stochastic process f such that M_t can be represented as*

$$M_t = \mathbb{E}_{\mathbb{P}}[M_0] + \int_0^t f(s, \omega) dW_s \quad a.s. \quad \forall t \geq 0. \quad (1.9)$$

In other words, every martingale with a finite second moment can be uniquely represented as a sum of its expected value at $t = 0$ and an Itô integral. For a trivial example with $M_t = W_t$, one has $f(t, \omega) = 1$. The martingale representation theorem is a useful result in finance for establishing hedging strategies.

1.3 Change of Measure and Girsanov's Theorem

Consider the filtered probability space $(\Omega, \mathcal{F}, \{F_t\}_{t \geq 0}, \mathbb{P})$. A probability measure \mathbb{Q} on \mathcal{F}_T is said to be *absolutely continuous* with respect to $\mathbb{P}|_{\mathcal{F}_T}$, if

$$P(A) = 0 \Rightarrow \mathbb{Q}(A), \quad \forall A \in \mathcal{F}_T.$$

The *Radon-Nikodym theorem* states that this is equivalent to the existence of a nonnegative \mathcal{F}_T -measurable random variable Z_T satisfying

$$d\mathbb{Q} = Z_T d\mathbb{P} \quad \text{on } \mathcal{F}_T. \quad (1.10)$$

Since $\mathbb{Q} \ll \mathbb{P}|_{\mathcal{F}_T}$ and the filtration \mathcal{F}_t is contained in \mathcal{F}_T for all $0 \leq t \leq T$, we also have that $\mathbb{Q}|_{\mathcal{F}_t} \ll \mathbb{P}|_{\mathcal{F}_t}$. It can then be shown that $Z_t := \frac{d(\mathbb{Q}|_{\mathcal{F}_t})}{d(\mathbb{P}|_{\mathcal{F}_t})}$ is a martingale with respect to \mathcal{F}_t and \mathbb{P} (see [13]). \mathbb{P} and \mathbb{Q} are said to be *equivalent* probability measures if and only if they are absolutely continuous to each other. They assign positive probabilities to the same events, and also agree which events are impossible. In pricing of assets as for example stocks in an arbitrage-free market, one moves from the physical measure \mathbb{P} to an equivalent risk-neutral measure \mathbb{Q} by applying Girsanov's theorem:

Theorem 1.3.1 (Girsanov's theorem). *Let $dX_t = \alpha(t, \omega)dt + \beta(t, \omega)dW_t$ under the probability measure \mathbb{P} . Assume that*

$$\mathbb{E}_{\mathbb{P}}[\exp(\frac{1}{2} \int_0^T \|\theta(t, \omega)\|^2 dt)] < \infty, \quad (1.11)$$

where

$$\theta(t, \omega) = \beta^{-1}(t, \omega)(\alpha(t, \omega) - \gamma(t, \omega)). \quad (1.12)$$

Then

$$M_t = \mathcal{E}(- \int_0^t \theta(s, \omega) dW_s) = \exp(- \int_0^t \theta(s, \omega) dW_s - \frac{1}{2} \int_0^t \theta^2(s, \omega) ds) \quad (1.13)$$

is a martingale under the equivalent measure \mathbb{Q} , defined by $d\mathbb{Q} = M_T d\mathbb{P}$, under which

$$W_t^* = W_t + \int_0^t \theta(s, \omega) ds \quad (1.14)$$

is a standard Brownian motion. X_t admits the integral representation

$$dX_t = \gamma(t, \omega)dt + \beta(t, \omega)dW_t^*. \quad (1.15)$$

Here, $\mathcal{E}(X)$ denotes the solution of the SDE $dY_t = Y_t dX_t$ with initial value $Y_0 = 1$. The Novikov condition (1.12) ensures that M_t is in fact a martingale.

Example 1.3.1 (Black-Scholes market). *By the change of measure*

$$M_t = \mathcal{E}(\int_0^t \frac{r - \mu}{\sigma} dW_s), \quad (1.16)$$

the stock price dynamics under \mathbb{Q} becomes

$$dS_t = \sigma S_t dW_t^*, \quad (1.17)$$

where

$$dW_t^* = dW_t + \frac{\mu - r}{\sigma} dt. \quad (1.18)$$

Since the Black-Scholes market is complete, there only exists one unique risk neutral measure. For incomplete and arbitrage-free markets, several or infinitely many risk neutral measures may exist, which in turn potentially give rise to a whole interval of arbitrage-free prices, rather than one unique price. This problem arises for example when modeling assets in terms of general jump processes, in place of Brownian motion. The latter is a special case of a family of stochastic processes called Lévy (or jump) processes.

1.4 Existence and Uniqueness of SDEs

The goal of this section is to shortly define when a given stochastic differential equation (SDE) has a solution and whether this solution is unique or not. Consider the SDE/initial value problem given by

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \quad \forall t \in [0, T], \quad X_0 = Z. \quad (1.19)$$

The following theorem gives the conditions for the existence and uniqueness of (1.19):

Theorem 1.4.1 (Existence and Uniqueness of SDEs). *[13] Let $T > 0$ and let $\mu : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^{n \times m}$ be measurable functions, for which there exist constants C and D such that*

$$|\mu(x, t)| + |\sigma(x, t)| \leq C(1 + |x|) \quad (1.20)$$

and

$$|\mu(x, t) - \mu(y, t)| + |\sigma(x, t) - \sigma(y, t)| \leq D|x - y|, \quad (1.21)$$

for all $t \in [0, T]$ and all $x, y \in \mathbb{R}^n$. Then the stochastic differential equation/initial value problem in (1.19) has a \mathbb{P} -almost surely unique t -continuous solution $(\omega, t) \rightarrow X_t(\omega)$, such that X_t is adapted to the filtration \mathcal{F}_t^Z generated by Z and $\{B_s\}_{s \leq t}$, and

$$\mathbb{E}_{\mathbb{P}}\left[\int_0^T |X_t^2| dt\right] < +\infty. \quad (1.22)$$

Condition (1.20) ensures the existence of a solution to (1.19) (the solution does not explode). (1.21) describes the uniqueness condition (also called the Lipschitz condition). If two t -continuous processes $X_t^1(t, \omega)$ and $X_t^2(t, \omega)$ satisfy both of these conditions and solve (1.19), then $X_t^1(t, \omega) = X_t^2(t, \omega)$ for all $t \leq T$ a.s.

Example 1.4.1. Consider the SDE

$$dX_t = \frac{1}{2}X_t dt + X_t dW_t, \quad (1.23)$$

i.e. $\mu(x, t) = \frac{1}{2}x$ and $\sigma(x, t) = x$. A solution of (1.23) exists, as condition (1.20) is satisfied:

$$|\mu(x, t)| + |\sigma(x, t)| = \left|\frac{1}{2}x\right| + |x| \leq \frac{3}{2}|x| \leq C(1 + |x|),$$

for any constant $C \geq \frac{3}{2}$. The solution X_t of (1.23) is unique, as condition (1.21) is satisfied:

$$|\mu(x, t) - \mu(y, t)| + |\sigma(x, t) - \sigma(y, t)| = \frac{1}{2}|x - y| + |x - y| \leq D|x - y|,$$

for any constant $D \geq \frac{3}{2}$. □

Chapter 2

Credit Risk and Credit Derivatives

The goal of this chapter is to define credit risk and to discuss pricing methods of the most common credit derivatives. It is based on material from [12], [6], [1], [8] and [9]. The resulting closed form expressions for the prices will mostly be given on a general form (with the exception of simple examples), as the next chapters will be concerned with deriving prices where the default risk is described in terms of specific stochastic models.

2.1 Credit Risk

Definition 2.1.1 (Credit risk [12]). *Credit risk is the risk that an obligor does not honour his obligations.*

In other words, credit risk is the risk that a debtor will fail to pay back his debt. Types of debts can be e.g. loans, bonds or mortgages. A loss (complete or partial) can arise when e.g. an insurance company is unable to pay a policy holder his obligation, or when a company cannot pay an employee his earned wages. Modeling default risk¹ is challenging, due to several factors. Defaults do not often occur, and they are difficult to predict. The losses are often large, and their exact sizes are not known until the date they happen. The probability of a default is in general very low, although it fluctuates considerably between different firms. There are companies that rank the creditworthiness of borrowers according to a standardized scale. As an example, Moody's use a scale from Aaa to C, where Aaa is the rating for the highest credit quality and C the lowest.

¹Unless otherwise stated, the terms “credit risk” and “default risk” are assumed mean the same thing

2.2 Credit Derivatives

Credit derivatives are tools for hedging (reducing) credit risk, that is, they allow trading of credit risk. The credit risk can be hedged by an investor by investing in a credit derivative. The risk is then transferred from the investor to the insurer in exchange for a fee. Credit derivatives can be traded speculatively and are negotiated privately, often called *over-the-counter* (OTC). In the case of pricing derivative securities, it is often assumed that the market is complete and arbitrage-free, and risk-neutral methods are utilized in order to derive fair prices. The question is whether these assumptions are valid for pricing credit derivatives as well.

One group of credit derivatives is the one where the payoff is only related to a default event, and excludes the second group of credit derivatives whose payoffs are dependent on the fluctuations in the credit quality of the underlying. The last group consists of credit derivatives that transfer the total risk of assets between two counterparties. There are three types of contracts, namely options, swaps and forward contracts.

2.2.1 Single-name Credit-risky Assets

Bonds

A bond is an investment in debt where money is loaned out to an entity. Of course, this is not done for free. It is usually arranged so that the investor regularly receives interest from the borrower up to the date of maturity of the agreement. The interest paid by the bond is called a *coupon*. As the contract expires, the borrower has promised to pay back the same amount he received in the first place, unless he defaults. The greater this risk of default is, the more expensive the loan should be made, often by charging higher interest. *Treasury bonds*, also called *treasuries*, are the safest bonds. They are issued by the U.S. government, and are considered risk-free, hence why they also pay a lower yield compared to other bonds. The most common type of bond is fixed-coupon bonds, where the coupons are determined in advance. *Zero-coupon bonds* (ZCBs) are bonds without coupons, and are bought on a deep discount. They are important as building blocks for modeling of credit risk, but are not often seen in corporate bond markets. In the following, pricing methods for ZCBs will be investigated further.

Pricing of ZCBs

The price of a bond refers to its expected discounted future payoffs. Consider a default-free ZCB maturing at time T , with face value 1. Assume a risk-free short-rate process r_t (that is, the interest rate expected from a risk-free investment). At time $t \leq T$, the bond has the price given by

$$B(t, T) = \mathbb{E}_{\mathbb{Q}}[e^{-r_t} \cdot \dots \cdot e^{-r_T} | \mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[e^{-(r_t + \dots + r_T)} | \mathcal{F}_t]. \quad (2.1)$$

The expectation is taken with respect to a risk-neutral probability \mathbb{Q} , conditioning upon the information \mathcal{F}_t available up to time t . This expression can be extended to hold in continuous time:

$$B(t, T) = \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^T r_u du} | \mathcal{F}_t]. \quad (2.2)$$

The outcome will now depend on which way one models the interest rate. An alternative is the simple assumption of a constant or deterministic function. However, a more realistic model takes into account the stochastic nature of the interest rate, by modeling it in terms of e.g. a *Vasicek* or *Cox, Ingersoll and Ross (CIR) model*. The models share the property of admitting closed form expressions for the bond prices.

In the case of a defaultable zero-coupon bond, there is a probability of default of the bond issuer. The price is now thus depending upon the default intensity, which lowers the price, due to a reduction in the expected discounted payoff from the bond. Let's assume that the defaultable zero-coupon bond pays the *recovery rate* δ if there is a default before T . The recovery rate will here be assumed to be a fraction of the face value. The price at time t of the bond now becomes

$$B^d(t, T) = \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^T r_u du} \mathbb{1}\{\tau > T\} + \delta e^{-\int_t^T r_u du} \mathbb{1}\{\tau \leq T\} | \mathcal{F}_t]. \quad (2.3)$$

As we will see later, there are several different ways to compute the default probabilities.

Credit spreads of bonds

The difference between the yield of a defaultable bond, $Y^d(t, T)$, and that of an equivalent default-free bond, $Y(t, T)$, is called the *credit spread* of the defaultable bond:

$$S(t, T) = Y^d(t, T) - Y(t, T). \quad (2.4)$$

In short, it reflects the additional yield the investor can make by investing in the defaultable bond compared to the default-free bond.

Asset swap

A holder of a bond could be interested in swapping the fixed coupons into a floating rate coupon (typically Euribor or LIBOR rate) plus a fixed spread. The fixed spread is called the asset swap spread. This swap in itself is called an asset swap.

Credit Default Swaps (CDS)

In recent years, the market for CDS has been growing rapidly. A CDS is an example of

one of the most common credit derivatives. Consider a pension fund that wishes to lend company A the amount L . In return, the company will pay back interest r on the loan until the end of the contract. The loan sum is then paid back to the pension fund, unless the company defaults. The pension fund can be insured against this default risk by investing in a default swap contract with another company B (with higher credit quality). This can be done by for example letting company B receive a fraction r' of the interest rate paid from company A until the contract expires. If a default occurs, the payments stop and company B compensates the pension fund with L .

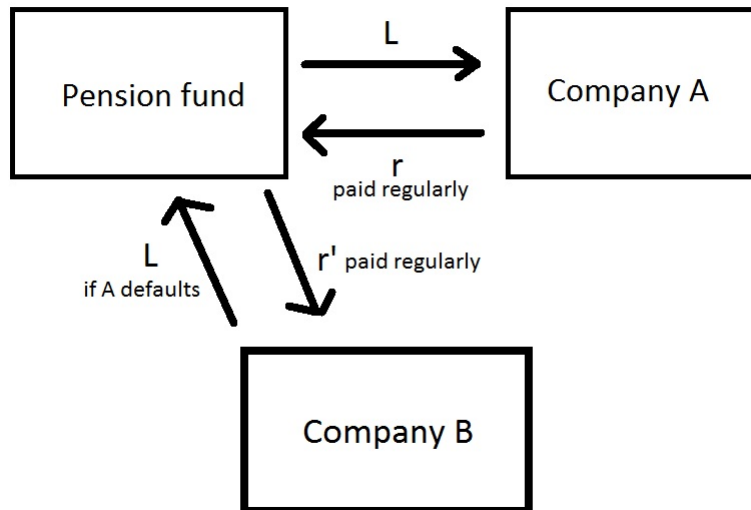


Figure 1.1: *Example of cash flows in a CDS contract.*

Pricing

In order to price credit default swaps, one looks at the expected discounted values of the payment streams of the protection buyer and insurer separately. The premium is then found by equating the two expressions. The interest rate will here be assumed to be stochastic and independent of the recovery rate and the default intensity. The payment streams will be on the following form:

Protection buyer: Assume that the protection buyer wants to insure one unit of money. He then pays the premium s at each time point $0 = t_0 < t_1 < \dots < t_n = T$ until maturity T of the contract, or stops if a default occurs. Let the default time be denoted by τ . His expected discounted cash flow becomes

$$EDPB := \mathbb{E}_{\mathbb{Q}}[s \sum_{i=0}^n e^{-\int_0^{t_i} r_u du} \mathbb{1}_{\{\tau > t_i\}}] = s \sum_{i=0}^n d(t_i) e(t_i), \quad (2.5)$$

where $d(t_i)$ denotes the expected discount factor from the beginning of the contract until the i -th payment date, while $e(t_i)$ is the survival probability up to time t_i .

Insurer: The insurer simply pays $1 - \delta$ if a default occurs before T , the expected discounted value of this payment is thus

$$EDI := \mathbb{E}_{\mathbb{Q}}[e^{-\int_0^T r_u du} (1 - \delta) \mathbb{1}_{\{\tau \leq T\}}] = (1 - \delta) \int_0^T d(u) (-de(u)). \quad (2.6)$$

Solving for the premium s , yields

$$s = \frac{(1 - \delta) \int_0^T d(u) (-de(u))}{\sum_{i=0}^n d(t_i) e(t_i)}. \quad (2.7)$$

How are such expression evaluated? It will depend on the model we use for the default probabilities. We define the default time by

$$\tau := \inf\{t > 0 : N_t = 1\}, \quad (2.8)$$

where N_t is a Poisson process with intensity γ_t . The default time is then the first time N_t jumps. In its most general form, the default intensity γ_t is a stochastic process. Let's first for the sake of simplicity look at the case where it is constant $\gamma_t = \gamma$ for all t and assume that the premiums are paid continuously. The survival probability becomes

$$e(t_i) = \mathbb{Q}(\tau > t_i) = e^{-\gamma t_i}. \quad (2.9)$$

Hence,

$$d(e(u)) = d(e^{-\gamma u}) = -\gamma e^{-\gamma u} du. \quad (2.10)$$

If the premiums are paid continuously, the sum in the denominator of (2.7) becomes an integral and cancels out with the integral in the numerator. We are left with

$$s = (1 - \delta)\gamma, \quad (2.11)$$

often referred to as the *credit triangle*. In the case of a deterministic default intensity $\gamma_t = \gamma(t)$, the survival probability becomes

$$e(t_i) = \mathbb{Q}(\tau > t_i) = e^{-\int_0^{t_i} \gamma(u) du}, \quad (2.12)$$

where the default time is the first time an inhomogeneous Poisson process with intensity $\gamma(t)$ jumps. Then,

$$d(e(u)) = d(e^{-\int_0^u \gamma(s) ds}) = -\gamma(u) e^{-\int_0^u \gamma(s) ds} du = -\gamma(u) e(u) du, \quad (2.13)$$

and the premium is given by

$$s = \frac{(1 - \delta) \int_0^T d(u) \gamma(u) e(u) du}{\int_0^T d(u) e(u) du}. \quad (2.14)$$

Calibration

Let $s_q(T_i)$ describe the quoted spreads of a certain CDS maturing at T_i , $i = 1, \dots, M$. [11] calibrates the CDS term structure by minimizing the root mean squared distance between the market spreads and the theoretical spreads $s_{th}(T_i)$,

$$\sqrt{\sum_{i=1}^M \frac{(s_q(T_i) - s_{th}(T_i))^2}{M}}, \quad (2.15)$$

with respect to the model parameters. They consider Ornstein-Uhlenbeck processes driven by a range of different jump processes as models for the default intensity. The implied survival curves are then constructed by bootstrapping methods, yielding the theoretical prices.

2.2.2 Portfolio Credit Derivatives

Collateralized Debt Obligations (CDOs)

A collateralized debt obligation (CDO) is a credit derivative that is backed by a pool of assets. It belongs to the group of so-called asset-backed securities. One example is a bank that gives out mortgages, car loans etc. The loans are then sold to an investment bank, where the loans are repackaged in tranches with respect to their risk levels. The tranches are sold to investors. The investors will then receive the principal payments plus interest rate (called the *collateral*). There is a risk that one or more loans will default before full repayment. The tranches with highest seniority will be paid first, while tranches with lower seniority will only be backed if there are funds left after covering the more senior tranches. The tranches with lower seniority are then riskier to invest in and therefore offer a higher interest rate to attract investors. Banks create these securitized assets in order to reduce credit exposure. It removes risky assets from their balance sheets, which in turn is lowering their capital requirements and allowing them to invest in new loans. CDO's are usually divided into two different types; *Collateralized loan obligations* (as in the example above) and *collateralized bond obligations* (where the asset pool consists of bonds). There are also *arbitrage CDOs*, where one sells tranches with profit. *Synthetic CDOs* are CDOs where the pool consists of CDS and not actual assets.

Pricing of CDOs

The pricing of CDOs is based on their cash flows. Assume we have I individual companies

and J tranches. Denote the accumulated portfolio loss up to t by L_t . We assume that the recovery rates δ_i for each company are identical, i.e. $\delta_i = \delta$. We give the portfolio the weights $\frac{1}{I}$ for each company. The loss then becomes

$$L_t = \frac{1-\delta}{I} \sum_{i=1}^I \mathbb{1}\{\tau^i \leq t\}. \quad (2.16)$$

Here τ^i is the default time of company i . Hence for each company that defaults, the loss is $\frac{1-\delta}{I}$, and L_t is the aggregate loss.

Assume that the face value of the assets is 1, and that it is divided in the J tranches. The individual tranches are assigned boundaries, called attachment points. The higher seniority of the tranche, the higher percentage of the assets they are entitled to in the event of a default, to cover their losses. The accumulated portfolio loss and the degree of seniority influence their individual losses, L_t^j :

$$L_t^j = \min(\max(0, L_t - l^j), u^j - l^j), \quad (2.17)$$

where l^j and u^j are the lower and upper boundaries for tranche j respectively.

The pricing of CDOs consists of determining the j -th spread s^j for the j -th tranche. This is done by calculating the expected discounted cash flows for the premium and default legs individually, then equating them and solving for s^j . s^j is called the fair spread for tranche j . Assume that the payments of the premium takes place at the time points $t_1 < \dots < t_n$. For tranche j , a premium is paid at each payment time t_k . The premium is a product of the fair spread s^j , what is remaining of the nominal of that tranche (i.e. $u^j - l^j - L_{t_k}^j$ and the time of the last period δ_{t_k} , hence $s^j(u^j - l^j - L_{t_k}^j)\delta_{t_k}$. The expected discounted value of this cash flow is then given by

$$\text{EDPL}^j := \sum_{k=1}^n e^{-rt_k} s^j (u^j - l^j - \mathbb{E}_{\mathbb{Q}}[L_{t_k}^j]) \Delta_{t_k}. \quad (2.18)$$

The losses of tranche j of the last period, $L_{t_k}^j - L_{t_{k-1}}^j$ are paid at each time t_k . The expected discounted value of this cash flow is

$$\text{EDDL}^j := \sum_{k=1}^n \mathbb{E}_{\mathbb{Q}}[L_{t_k}^j] - \mathbb{E}_{\mathbb{Q}}[L_{t_{k-1}}^j]. \quad (2.19)$$

The fair spread value of tranche j is then given by

$$s^j = \frac{\sum_{k=1}^n (\mathbb{E}_{\mathbb{Q}}[L_{t_k}^j] - \mathbb{E}_{\mathbb{Q}}[L_{t_{k-1}}^j])}{\sum_{k=1}^n e^{-rt_k} (u^j - l^j - \mathbb{E}_{\mathbb{Q}}[L_{t_k}^j]) \Delta_{t_k}}. \quad (2.20)$$

Index (Portfolio) CDS

Similarly to the case of a single-name credit derivative, the protection buyer regularly pays the protection seller a premium S_T until maturity T of the contract, or until a default on the reference portfolio occurs. The protection seller then pays the protection buyer a compensation if the reference portfolio defaults before T . Assume the portfolio consists of l assets, and N_i denotes the face value of asset i . The portfolio face value then becomes $N = N_1 + \dots + N_l$.

Pricing

Pricing is based on the expected discounted cash flows of the two counterparts. The portfolio-loss at each time point $t \in [0, T]$ is given by

$$L_t = \sum_{i=1}^l (1 - R^i) \mathbb{1}_{\{\tau_i \leq t\}}, \quad (2.21)$$

where R^i is the recovery rate. What remains after the potential losses, is simply the initial value of the portfolio minus the loss:

$$N_t = N_0 - L_t. \quad (2.22)$$

The expected discounted cash flow of the protection buyer, is given by

$$\text{EDPL} = \mathbb{E}_{\mathbb{Q}} \left[\sum_{k=1}^n e^{-rt_k} S_T \Delta t_k N_{t_k} \right], \quad (2.23)$$

whereas the expected discounted cash flow of the insurer becomes

$$\text{EDI} = \mathbb{E}_{\mathbb{Q}} \left[\sum_{k=1}^n e^{-rt_k} (L_{t_k} - L_{t_{k-1}}) \right]. \quad (2.24)$$

The fair value of the premium is found by equating (2.23) and (2.24).

n-th to Default Contracts (Basket)

Consider again a portfolio with l credit-risky assets. In this case, the protection buyer keeps on paying premium $s^{(n)}$ as long as the number of defaults in the portfolio is not exceeding a number $n \in \{1, \dots, l\}$. The insurer will then compensate the protection buyer if n defaults occur before maturity of the contract.

Pricing

As above, pricing is based on the expected discounted values of the cash flows, which now will depend on the distribution of the default times

$$0 \leq \tau_{(1)} \leq \dots \leq \tau_{(n)} \leq \dots \leq \tau_{(l)}.$$

With the assumption that the portfolio face value is $N = 1$, the expected discounted cash flow of the protection buyer is

$$\text{EDPL}^{(n)} = \mathbb{E}_{\mathbb{Q}} \left[\sum_{k=1}^n e^{-rt_k} s^{(n)} \Delta t_k \mathbb{1}_{\{\tau_{(n)} > t_k\}}, \right] \quad (2.25)$$

while for the insurer

$$\text{EDI}^{(n)} \mathbb{E}_{\mathbb{Q}} [(1 - R) e^{-r\tau_{(n)}} \mathbb{1}_{\{0 \leq \tau_{(n)} \leq T\}}]. \quad (2.26)$$

$s^{(n)}$ is then calculated by equating (2.25) and (2.26).

Mutually Independent Defaults

This case is best illustrated by an example:

Example 2.2.1 (Digital default put of basket type). *Consider a portfolio consisting of n defaultable assets, whose default times τ_1, \dots, τ_n are mutually independent and admits their intensities $\gamma_1(t), \dots, \gamma_n(t)$. Each asset has the cumulative distribution function $F_i(t) = \mathbb{Q}(\tau_i \leq t) = 1 - e^{-\int_0^t \gamma_i(s) ds}$. Let $\tau_{(i)}$ denote the time of the i -th default in the portfolio, whereas $F_{(i)}(t) = \mathbb{Q}(\tau_{(i)} \leq t)$. Assume a deterministic interest rate and that the contract pays one unit of cash if i assets default before (or at) maturity T . Its value at $t = 0$ becomes*

$$S_0 = \mathbb{E}_{\mathbb{Q}} [B_{\tau}^{-1} \mathbb{1}_{\{\tau_{(i)} \leq T\}}] = \int_{(0,T]} B_u^{-1} dF_i(u) = \int_0^T B_u^{-1} \gamma_{(i)}(u) e^{-\int_0^u \gamma_i(s) ds} du, \quad (2.27)$$

where $\gamma_{(i)}$ is the combined intensity of the i -th first defaults.

Modeling by Copulas

The assumption of independence between the default times is not a very realistic one. With a so-called *copula function* $C : [0, 1]^n \rightarrow [0, 1]$, their dependence can be taken into account. It expresses the cumulative multivariate distribution of the default times, with a given correlation structure, in terms of their marginal probability distributions:

$$\mathbb{Q}(\tau_1 \leq t, \dots, \tau_n \leq t) = C(F_1(t_1), \dots, F_n(t_n)) \quad (2.28)$$

Given a static model, with fixed time horizon $[0, T]$, the copula function admits the basic properties

1. Probability of no defaults:

$$\mathbb{Q}(u_i \leq F_i(t), \forall i \leq l) = C(F_1(t), \dots, F_n(t))$$

2. Probability of no default for the k first assets

$$\mathbb{Q}(u_i \leq F_i(t) \forall i \leq k) = C(F_1(t), \dots, F_k(t), 1, \dots, 1)$$

3. No default within a set S of assets:

$$C(u_1, \dots, u_l)$$

where $u_i = F_i(t)$ if $i \in S$ and 1 otherwise.

Chapter 3

Structural Models

A central part of pricing credit derivatives is finding an appropriate model for the credit risk. In general, there are two types of models:

- *Structural models*, where one models the value of the firm's assets. We will look at the so-called *Merton model* and some of its extensions.
- *Intensity based models*, where one is interested in modeling the factors that may influence a default event, but usually not what exactly triggers it.

This chapter is based on [8], [9] and [1].

3.1 The Merton model

This is an application of Black & Scholes' option pricing model to corporate debt. The idea is that one models the value of a firm, and defines it to default if the value of its assets falls below the value of its liabilities. A default is here only possible at maturity. The assumption is that we are in a standard Black & Scholes market. This implies the properties of a frictionless market, that is, there are no transaction costs. Borrowing or lending is done through a money market account with a constant risk free rate r . The discount factor is thus given by $B(t, T) = e^{-r(T-t)}$. We are considering the filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q}, \{\mathcal{F}_t\})$, where \mathbb{Q} is the spot martingale measure. The firm value is modeled in terms of a stochastic process, $V_t = E(V_t) + D(V_t)$, where $E(V_t)$ and $D(V_t)$ are the values of the equity and debt, respectively. The debt here is a defaultable zero-coupon bond with face value D , maturing at T . \mathcal{F}_t is the σ -algebra generated by V_t . If the firm value drops below a boundary d , the firm defaults. The dynamics of V_t is given by

$$dV_t = (r - \kappa)V_t dt + \sigma V_t dW_t, \quad (3.1)$$

where W_t is geometric Brownian motion and σ is the constant volatility of V_t . κ describes for nonnegative values a payout from the firm, while negative values means that one has an inflow of capital.

At maturity T , the payoff to the bond holder is

$$D(V_T) = \mathbb{1}_{\{\tau > T\}}D + \mathbb{1}_{\{\tau \leq T\}}V_T = \min(D, V_T) = D - \max(D - V_T, 0), \quad (3.2)$$

which is the difference between the face value of the bond and the payoff from a put option on the firm value V_T exercised at T . The value for all t is thus given by

$$D(V_t) = B^d(t, T) = e^{-r(T-t)}D - P_t, \quad (3.3)$$

where P_t denotes the time t price of the put.

Similarly for the equity, the payoff at T is

$$E(V_T) = V_T - \min(V_T, D) = \max(V_T - D, 0), \quad (3.4)$$

which is a call option on the firm value with strike D . Its value for all t is thus the price C_t of the option:

$$E(V_t) = V_t - D(V_t) = V_t - De^{-r(T-t)} + P_t = C_t, \quad (3.5)$$

following the put-call parity.

The well-known formulas for the time t prices are applied. The defaultable bond price for $t \in [0, T]$ is thus given by

$$B^d(t, T) = E(V_t) = V_t e^{-\kappa(T-t)} \Phi(-d_1(V_t, T-t)) + De^{-r(T-t)} \Phi(d_2(V_t, T-t)), \quad (3.6)$$

where

$$d_1(V_t, T-t) = \frac{\ln(\frac{V_t}{D}) + (r - \kappa + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad (3.7)$$

and

$$d_2(V_t, T-t) = \frac{\ln(\frac{V_t}{D}) + (r - \kappa - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \quad (3.8)$$

$\Phi(d_2)$ is the probability of exercising the call option (i.e. the probability of no default), $\Phi(-d_2)$ is thus the default probability.

3.2 Hedging in the Merton model

The corresponding unique replicating strategy for the Merton model, is through results from the Black-Scholes model, given by

Corollary 3.2.1 (Replicating strategy in the Merton model). [1] The unique replicating strategy for a defaultable bond involves holding at any time $t \leq T$ the $\phi_t^1 V_t$ units of cash invested in the firm's value and $\phi_t^2 e^{-r(T-t)}$ units of cash invested in default-free bonds, where for every $t \in [0, T]$

$$\phi_t^1 = e^{-\kappa(T-t)} \Phi(-d_1(V_t, T-t)) \quad (3.9)$$

and

$$\phi_t^2 = D\Phi(d_2(V_t, T-t)). \quad (3.10)$$

3.3 Credit Spreads in the Merton model

For the credit spreads in the Merton model, it can be shown that

$$\lim_{t \rightarrow T} S(t, T) = \begin{cases} \infty, & \text{if } V_T < D. \\ 0, & \text{if } V_T \geq D. \end{cases} \quad (3.11)$$

The main drawback of the structural approach is that it tends to underestimate risk, as τ is a predictable stopping time w.r.t. the filtration generated by Brownian motion. Additionally, the short-term credit spreads for a firm goes to zero if it is close to a default. Empirical data (see e.g. Jones et. al. 1984) contradicts this fact.

3.4 Extensions of the Merton model

It's not very realistic to have the possibility of a default at maturity only. Black and Cox (1976) defined instead the default time as the *first* time the firm value hits the boundary d :

$$\tau := \inf\{t > 0 : V_t \leq d\}. \quad (3.12)$$

This is called a *first-passage time* model. It is possible to calculate the distribution of $\min_{0 \leq s \leq t} V_s$, which again can be used to find the default probability and thus also prices. Duffie and Lando (2001) found a way to incorporate unexpected defaults by using a different filtration \mathbb{F} . The firm value is here assumed to be observable only at certain time points. In addition, they corrected for incomplete accounting information by using a Gaussian

random variable to disturb the observations. Fioriani, Luciano and Semeraro [7] calibrated the Merton model including a pure-jump process, more specifically a *Variance Gamma* (VG) process, and showed that this corrected for under/overprediction of the low/high risk credit spreads. Cariboni and Schoutens [11] showed that with this method, the credit spreads become positive also for short maturities. However, the distribution of the first-passage times becomes unknown, resulting in bond and CDS prices becoming unavailable in closed form.

Chapter 4

Intensity Based Models

This is the most popular model for pricing credit derivatives and credit risk. It is reasonably simple to calibrate such models to market data. This chapter is based on material from [2], [8] and [1]. We begin by introducing the concepts of Hazard processes and random times. The cash flows of general defaultable claims are then described in detail. Finally, trading strategies and hedging methods in a defaultable market are discussed in a simplified setting.

4.1 Hazard Processes and Random Times

Consider the probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$ equipped with the filtration $\mathbb{F} = (\mathcal{F}_t)$. On this probability space, the default time τ is considered a nonnegative random variable. Introduce the default (jump) process $H_t = \mathbb{1}_{\{\tau \leq t\}}$, which generates the σ -algebra \mathcal{H}_t . Its filtration is then $\mathbb{H} = \sigma\{H_u : u \leq t\}$. $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ is an enlarged filtration, where $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$. \mathbb{G} thus contains the information about the default event.

The *default process* F_t denotes the probability of a default prior to time t , given the information \mathcal{F}_t up to time t :

$$F_t = \mathbb{Q}^*(\tau \leq t | \mathcal{F}_t) \quad (4.1)$$

The corresponding "survival" (i.e. no default) function G_t is then given by

$$G_t = 1 - F_t = \mathbb{Q}^*(\tau > t | \mathcal{F}_t) \quad (4.2)$$

The so-called \mathbb{F} -*hazard process* of τ under \mathbb{Q}^* is defined by

$$\Gamma_t = -\log G_t = -\log(1 - F_t), \quad \forall t \in \mathbb{R}_+. \quad (4.3)$$

4.2 Defaultable Claims

A general defaultable claim maturing at T will be denoted by the quadruple (τ, X, R, A) , where

- τ denotes the default time of the defaultable claim.
- X is the promised payoff to the claim holder, given that a default has not occurred prior to (or at) maturity.
- R describes the amount the claim owner receives if a default happens before (or at) maturity.
- A describes the promised dividends received by the claim holder, should a default occur before (or at) maturity.

The following assumptions will be made:

1. The default intensity γ_t is the solution of the Vasicek model driven by Brownian motion W_t .
2. The hazard process Γ is given by $\Gamma_t = \int_0^t \gamma_u du$ (and is thus continuous).
3. X is \mathcal{F}_T -measurable, that is, its value becomes known at maturity.
4. R is an \mathbb{F} -predictable bounded process.
5. A is an \mathbb{F} -predictable bounded process of finite variation.
6. The interest rate r_t will follow an \mathbb{F} -progressively measurable process, such that the savings account, B_t is given by

$$B_t = \exp\left(\int_0^t r_u du\right), \quad \forall t \in \mathbb{R}_+. \quad (4.4)$$

4.2.1 Cash Flows and Risk-neutral Valuation of a Defaultable Claim

In order to describe all the cash flows associated with (τ, X, R, A) , the *dividend-process* D is defined as

$$D_t = X \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{[T, \infty)(t)} + \int_{(0, T]} (1 - H_u) dA_u + \int_{(0, T]} R_u dH_u. \quad (4.5)$$

Definition 4.2.1 (Ex-dividend price process of a defaultable claim). [2] The ex-dividend price process E_t , for $t < T$, is given by

$$E_t = B_t \mathbb{E}_{\mathbb{Q}^*} \left[\int_{(t,T)} B_u^{-1} dD_u | \mathcal{G}_t \right], \quad (4.6)$$

where B_t denotes the savings account and \mathbb{Q}^* is the spot martingale measure. At T ,

$$E_T = X \mathbb{1}_{\{\tau > T\}} + R_T \mathbb{1}_{\{\tau \leq T\}}. \quad (4.7)$$

It is clear that for all t , (4.6) can be written as $E_t = E_t[X] + E_t[R] + E_t[A]$. From now on, $E_t = S_t^0$, where the latter is called the *pre-default value* of the claim. Each of the three terms will now be discussed separately. We will make use of the following well-known results:

Lemma 4.2.1. [1] Let X be both a \mathcal{G} -measurable and \mathcal{F}_T -measurable, integrable random variable. Then for $t \leq T$,

$$\mathbb{E}_{\mathbb{Q}^*} [\mathbb{1}_{\{\tau > T\}} X | \mathcal{G}_t] = \mathbb{1}_{\{\tau > T\}} \mathbb{E}_{\mathbb{Q}^*} [e^{\Gamma_t - \Gamma_T} X | \mathcal{F}_t]. \quad (4.8)$$

Lemma 4.2.2. [1] Let $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ be bounded and continuous. Then

$$\mathbb{E}_{\mathbb{Q}^*} [\mathbb{1}_{\{t \leq \tau \leq T\}} h(\tau) | \mathcal{G}_t] = \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E}_{\mathbb{Q}^*} \left[\int_{(t,T)} h(u) dF_u | \mathcal{F}_t \right]. \quad (4.9)$$

Promised payoff

Let's first consider the price of the promised payoff X , i.e. $E_t[X]$. Lemma 4.2.1 gives

$$\begin{aligned} E_t(X) &= B_t \mathbb{E}_{\mathbb{Q}^*} [B_T^{-1} \mathbb{1}_{\{\tau > T\}} X | \mathcal{G}_t] = \mathbb{1}_{\{\tau > T\}} B_t \mathbb{E}_{\mathbb{Q}^*} [B_T^{-1} e^{\Gamma_t - \Gamma_T} X | \mathcal{F}_t] = \\ &= \mathbb{1}_{\{\tau > T\}} B_t e^{\int_0^t \gamma_u du} \mathbb{E}_{\mathbb{Q}^*} [e^{-\int_0^T \gamma_u + r_u du} X | \mathcal{F}_t] = \\ &= \mathbb{1}_{\{\tau > T\}} \tilde{B}_t \mathbb{E}_{\mathbb{Q}^*} [\tilde{B}_T^{-1} X | \mathcal{F}_t] = \mathbb{1}_{\{\tau > T\}} \tilde{E}_t[X], \end{aligned} \quad (4.10)$$

where $\tilde{B}_t := e^{\int_0^t \gamma_u + r_u du}$ is referred to as the *default-risk adjusted savings account*.

Recovery payoff

Next is the price of the recovery R . By Lemma 4.2.2 with $h(\tau) = B_\tau^{-1} R_\tau$ and $dF_u = \gamma_u e^{\int_0^u \gamma_v dv} du$:

$$\begin{aligned} E_t[R] &= B_t \mathbb{E}_{\mathbb{Q}^*} [B_\tau^{-1} \mathbb{1}_{\{t < \tau \leq T\}} R_\tau | \mathcal{G}_t] = \mathbb{1}_{\{\tau > T\}} B_t \mathbb{E}_{\mathbb{Q}^*} \left[\int_t^T B_u^{-1} R_u e^{\int_0^u \gamma_v dv} du | \mathcal{F}_t \right] \\ &= \mathbb{1}_{\{\tau > T\}} \mathbb{E}_{\mathbb{Q}^*} \left[\int_t^T R_u e^{-\int_t^u r_v + \gamma_v dv} \gamma_u du | \mathcal{F}_t \right] = \mathbb{1}_{\{\tau > T\}} \tilde{E}_t[R]. \end{aligned} \quad (4.11)$$

Dividend payoff

$$\begin{aligned}
E_t[A] &= B_t \mathbb{E}_{\mathbb{Q}^*} \left[\int_{(t,T]} B_u^{-1} (1 - dH_u) dA_u | \mathcal{G}_t \right] = \mathbb{1}_{\{\tau > T\}} B_t \mathbb{E}_{\mathbb{Q}^*} \left[\int_{(t,T]} B_u^{-1} e^{\Gamma_t - \Gamma_u} dA_u | \mathcal{F}_t \right] \\
&= \mathbb{1}_{\{\tau > T\}} \mathbb{E}_{\mathbb{Q}^*} \left[\int_{(t,T]} e^{-\int_t^u r_v + \gamma_v dv} dA_u | \mathcal{F}_t \right] = \mathbb{1}_{\{\tau > T\}} \tilde{E}_t[A].
\end{aligned} \tag{4.12}$$

The terms $\tilde{E}_t[X]$, $\tilde{E}_t[R]$ and $\tilde{E}_t[A]$ are the *pre-default values* of the promised payoff, recovery and promised dividends, respectively. In general, calculating these expressions is a non-trivial task. In summary,

$$S_t^0 = \mathbb{1}_{\{\tau > t\}} \mathbb{E}_{\mathbb{Q}^*} \left[\int_{(t,T]} e^{-\int_t^u r_v + \gamma_v dv} (dA_u + \gamma_u R_u du) + X e^{-\int_t^T r_v + \gamma_v dv} | \mathcal{F}_t \right]. \tag{4.13}$$

The simplest case: A defaultable ZCB.

Let's for simplicity consider the case of a defaultable ZCB, where

- Both the default intensity $\gamma(t)$ and the interest rate $r(t)$ are deterministic.
- There is zero recovery (i.e. $R=0$) and no promised dividends (i.e. $A=0$).
- The promised contingent claim X is now the face value 1.

The time t price of a default-free ZCB under a deterministic interest rate, maturing at T , is given by

$$B(t, T) = e^{-\int_t^T r(v) dv}, \quad \forall t \in [0, T]. \tag{4.14}$$

The pre-default value of the defaultable bond becomes (by (4.13)):

$$\begin{aligned}
S_t^0 &= \mathbb{1}_{\{\tau > t\}} \mathbb{E}_{\mathbb{Q}^*} [e^{-\int_t^T r(v) + \gamma(v) dv} | \mathcal{F}_t] = \mathbb{1}_{\{\tau > t\}} e^{-\int_t^T r(v) + \gamma(v) dv} = \\
&\quad \mathbb{1}_{\{\tau > t\}} B(t, T) e^{-\int_t^T \gamma(v) dv}.
\end{aligned} \tag{4.15}$$

Since the recovery is zero, the pre-default value is the value of the claim for all $t \in [0, T]$.

Including deterministic recovery and dividends

Assume now that the recovery and dividend processes R_t and A_t are given by continuous

functions $R, A : \mathbb{R}_+ \rightarrow \mathbb{R}$, respectively.

The pre-default value of the zero-coupon defaultable bond then becomes

$$\begin{aligned} S_t^0 &= \mathbb{1}_{\{\tau > t\}} \mathbb{E}_{\mathbb{Q}^*} \left[\int_t^T e^{-\int_t^u r(v) + \gamma(v) dv} (A(u) + \gamma(u) R(u)) du + e^{-\int_t^T r(v) + \gamma(v) dv} | \mathcal{F}_t \right] \\ &= \mathbb{1}_{\{\tau > t\}} \left\{ \int_t^T e^{-\int_t^u r(v) + \gamma(v) dv} (A(u) + \gamma(u) R(u)) du + B(t, T) e^{-\int_t^T \gamma(v) dv} \right\}, \end{aligned} \quad (4.16)$$

as there is still not any randomness involved. Notice that this is no longer the value of the bond for all t , due to the payout after default. The term $\mathbb{1}_{\{\tau \leq t\}} R(\tau) e^{\int_\tau^t r(v) dv}$ has to be added to S_t^0 to correct for this (it is the discounted value of the recovery function at τ). If one wants to use a fixed recovery rate δ instead, one can let $R(t) \equiv \delta$.

4.2.2 Trading Strategies

Consider a portfolio with the trading strategy $\phi_t = (\phi_t^1, \dots, \phi_t^k)$, consisting of m defaultable assets $\{Y^i\}_{i=1}^m$ and $k - m$ default-free assets $\{Y^i\}_{i=m+1}^k$, assumed to be continuous semimartingales (which can be extended to general semimartingales as jump processes). The default time τ is the same for all the defaultable assets: Once one of the defaultable assets defaults, every other defaultable asset defaults. The corresponding value process is then given by

$$V_t(\phi) = \sum_{i=1}^k \phi_t^i Y_t^i, \quad \forall t \in [0, T]. \quad (4.17)$$

A self-financing trading strategy for a defaultable claim is defined as follows:

Definition 4.2.2. [2] *The trading strategy ϕ_t is said to be self-financing, if*

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^m \int_0^t \phi_u^i dY_u^i + \sum_{i=m+1}^k \int_0^t \phi_u^i dY_u^i, \quad \forall t \in [0, T]. \quad (4.18)$$

We assumed that recovery is only paid at the time of default, we will therefore only be concerned with trading strategies prior to (or at) maturity. The time interval under consideration is thus given by

$$[[0, \tau \wedge T]] = \{(t, \omega) \in \mathbb{R}_+ \times \Omega : 0 \leq t \leq \tau(\omega) \wedge T\}. \quad (4.19)$$

This also explains why we only need to consider \mathbb{F} -predictable trading strategies (rather than \mathbb{G} -predictable), as we never deal with the trading strategy after a default has occurred.

Definition 4.2.3. [2] A trading strategy $\phi_t = (\phi_t^1, \dots, \phi_t^k)$ is an \mathbb{F} -predictable stochastic process.

A replicating strategy for a defaultable claim is a trading strategy ϕ_t such that the corresponding value process V_t matches the pre-default value of the defaultable claim at all times up to maturity/default, where it equals the payoff X (if no default prior to or at maturity), otherwise the recovery R_τ . For defaultable claims with no promised dividends, we have the following definition of a replicating strategy:

Definition 4.2.4 (Replicating strategy for a defaultable claim [2]). A self-financing trading strategy is said to be a replicating strategy for a defaultable claim $(\tau, X, 0, R)$ if and only if the following hold:

1. $V_t(\phi) = \tilde{E}_t(X) + \tilde{E}_t(R)$ on $[0, \tau \wedge T]$.
2. $V_T(\phi) = R_\tau$ on $\{\tau \leq T\}$.
3. $V_T(\phi) = X$ on $\{\tau \geq T\}$.

A defaultable claim is called attainable if it admits at least one replicating strategy.

4.2.3 Hedging of Defaultable Claims

The goal of this section is to replicate defaultable claims with continuous trading in a defaultable bond and default-free securities. For the sake of simplicity, we will consider the case where the defaultable claim has zero recovery and zero promised dividends, i.e. a defaultable claim on the form $(\tau, X, 0, 0)$.

Consider an arbitrage-free and complete market model for the default-free securities, over the time horizon $[0, T^*]$. That is, in this market, every contingent claim is attainable and there exists a unique martingale (pricing) measure \mathbb{P}^* . For this market we model the uncertainties in the securities through \mathbb{F} on the probability space $(\hat{\Omega}, \mathcal{F}, \mathbb{P})$, where \mathbb{P} is equivalent to \mathbb{P}^* on \mathcal{F}_{T^*} . It is assumed to exist an extended probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$, which includes also defaultable claims, priced under a martingale measure \mathbb{Q}^* . When \mathbb{Q}^* is restricted to \mathcal{F}_{T^*} , it coincides with \mathbb{P}^* , \mathbb{Q}^* prices thus both the default-free and defaultable claims.

The pre-default value of the defaultable claim $(\tau, X, 0, 0)$ is given by

$$S_t^0 = B_t \mathbb{E}_{\mathbb{Q}^*}[B_T^{-1} X \mathbb{1}_{\{\tau > T\}} | \mathcal{G}_t]. \quad (4.20)$$

Its discounted value process $\bar{S}_t^0 = \frac{S_t^0}{B_t}$, becomes

$$\bar{S}_t^0 = \mathbb{E}_{\mathbb{Q}^*}[B_T^{-1}X\mathbb{1}_{\{\tau>T\}}|\mathcal{G}_t] = L_t m_t^X, \quad (4.21)$$

where $L_t = \mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}$ and $m_t^X = \mathbb{E}_{\mathbb{Q}^*}[B_T^{-1}X\mathbb{1}_{\{\tau>T\}}|\mathcal{F}_t]$.

m_t^X is a \mathbb{G} -martingale with respect to \mathbb{Q}^* , and so is L_t by the following results:

Lemma 4.2.3. [1] *Let Y be \mathcal{G} -measurable. Assume the auxiliary filtration \mathbb{F} is given, such that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, i.e. $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ for any $t \in \mathbb{R}_+$. Then for $s \geq t$*

$$\mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\{\tau>s\}}Y|\mathcal{G}_t] = \mathbb{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\{\tau>s\}}e^{\Gamma_t}Y|\mathcal{F}_t]. \quad (4.22)$$

Lemma 4.2.4. [1] *Assume the auxiliary filtration \mathbb{F} is given, such that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, i.e. $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ for any $t \in \mathbb{R}_+$. Then*

$$L_t := \mathbb{1}_{\{\tau>t\}}e^{\Gamma_t} \quad (4.23)$$

is a \mathbb{G} -martingale.

Proof

We need to show that for $s \geq t$, we have $\mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\{\tau>s\}}e^{\Gamma_s}|\mathcal{G}_t] = \mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}$. Using Lemma (4.2.3), this translates to showing that with $Y = e^{\Gamma_t}$

$$\mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}\mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\{\tau>s\}}e^{\Gamma_s}|\mathcal{F}_t] = \mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}$$

i.e. $\mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\{\tau>s\}}e^{\Gamma_s}|\mathcal{F}_t] = 1$:

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\{\tau>s\}}e^{\Gamma_s}|\mathcal{F}_t] &= \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\{\tau>s\}}e^{\Gamma_s}|\mathcal{F}_s]|\mathcal{F}_t] = \\ \mathbb{E}_{\mathbb{P}}[e^{\Gamma_s}\mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_s]|\mathcal{F}_t] &= \mathbb{E}_{\mathbb{P}}[e^{\Gamma_s}e^{-\Gamma_s}|\mathcal{F}_t] = 1 \end{aligned}$$

□

Lemma 4.2.5. [1] *Let Γ be continuous and increasing. Then*

$$\hat{M}_t = H_t - \Gamma_{t \wedge \tau} \quad (4.24)$$

is a \mathbb{G} -martingale and solves

$$dL_t = -L_{t-}d\hat{M}_t. \quad (4.25)$$

$\bar{S}_t^0 = 0$ after a default with no recovery, hence \bar{S}_t^0 describes thus the discounted value of a defaultable claim for all $t \in [0, T]$, $T \leq T^*$.

Let $Y_1 = e^{-\Gamma_t}$ be the price process of a default-free claim. Its discounted price is an \mathbb{F} -martingale,

$$m_t = \mathbb{E}_{\mathbb{Q}^*}[B_T^{-1}Y_1|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}^*}[B_T^{-1}e^{-\Gamma_t}|\mathcal{F}_t] = \mathbb{E}_{\mathbb{P}^*}[B_T^{-1}e^{-\Gamma_t}|\mathcal{F}_t], \quad (4.26)$$

where the last equality follows from the assumption that \mathbb{Q}^* restricted to \mathcal{F}_{T^*} coincides with \mathbb{P}^* .

Lemma 4.2.6. [1] *The \mathbb{G} -martingale \bar{S}_t^0 admits the integral representation*

$$\bar{S}_t^0 = \bar{S}_0^0 + \int_0^{t \wedge \tau} e^{\Gamma_u} dm_u^X - \int_{(0, t \wedge \tau]} e^{\Gamma_u} m_u^X d\hat{M}_u. \quad (4.27)$$

The price process of a defaultable bond is given by

$$B^d(t, T) = B_t \mathbb{E}_{\mathbb{Q}^*}[B_T^{-1} \mathbb{1}_{\{\tau \leq t\}} | \mathcal{G}_t]. \quad (4.28)$$

Define $Z^0(t, T)$ as the discounted value of the defaultable bond, i.e. $Z^0(t, T) = \frac{B^d(t, T)}{B_t}$. By Itô's formula (1.2.1), one can find that

$$d(Z^0(t, T)) = L_{t-} dm_t - L_{t-} m_t d\hat{M}_t = L_{t-} dm_t - e^{-\Gamma_t} d\hat{M}_t. \quad (4.29)$$

Utilizing that

$$dZ^0(t, T) - L_{t-} dm_t = -e^{\Gamma_t} m_t d\hat{M}_t, \quad (4.30)$$

one obtains the representation

$$\bar{S}_t^0 = \bar{S}_0^0 + \int_0^{t \wedge \tau} e^{\Gamma_u} dm_u^X - \int_0^{t \wedge \tau} e^{\Gamma_u} m_u^X m_u^{-1} dm_u + \int_{(0, t \wedge \tau]} m_u^X m_u^{-1} dZ^0(u, T). \quad (4.31)$$

We are now ready to state the final result.

Proposition 4.2.1. [1] *Let us denote $\zeta_t^X = m_t^X m_t^{-1}$. On the set $\{t \leq \tau\}$, the replicating strategy for the discounted price process \bar{S}_t^0 equals*

$$\phi_t^0 = \zeta_t^X, \quad \phi_t^1 = e^{\Gamma_t} \zeta_t^X, \quad \phi_t^2 = e^{\Gamma_t}, \quad (4.32)$$

where the hedging instruments are: the discounted price process $Z^0(t, T)$ of the T -maturity defaultable zero-coupon bond with zero recovery and the discounted price processes of default-free claims $Y_1 = e^{\Gamma_t}$ and $Y_2 = X e^{\Gamma_t}$. On the set $\{t > \tau\}$, the replicating strategy is identically equal to zero.

In other words, $\phi_t = (\phi_t^0, \phi_t^1, \phi_t^2) = (\zeta_t^X, e^{\Gamma_t} \zeta_t^X, e^{\Gamma_t})$ describes the number of units to hold at all times t in order to replicate the defaultable claim $(\tau, X, 0, 0)$, of the defaultable zero-coupon bond and the two default-free claims respectively.

Chapter 5

Affine Intensity Models and ZCB Pricing

This chapter will discuss how to price both default-free and defaultable ZCBs where the interest rate and default intensity are given in terms of the two most popular stochastic models for modeling interest rates. It is based on material from [3], [8] and [10].

5.1 Affine Term Structure

Consider a default-free ZCB $B(t, T)$ maturing at T . Assume that the stochastic interest rate r_t satisfies the SDE

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t. \quad (5.1)$$

The interest rate is said to have an *affine term structure*, if

$$B(t, T) = \exp(A(t, T) - C(t, T)r_t), \quad (5.2)$$

where A and B are deterministic functions. We will now consider two types of such models, namely the *Cox, Ingersoll and Ross (CIR) model* and the *Vasicek model*, and derive their formulas for the ZCB prices.

5.2 The Multifactor CIR Model

Let the dynamics of the stochastic process γ_t be given by the CIR model, i.e.

$$d\gamma_t = (\alpha - \beta\gamma_t)dt + \delta\sqrt{\gamma_t}dW_t, \quad (5.3)$$

where α , β and δ are constants. The requirement $2\alpha \geq \delta^2$ ensures positivity of γ at all times. (5.3) is more commonly written as

$$d\gamma_t = k(\theta - \gamma_t)dt + \delta\sqrt{\gamma_t}dW_t, \quad (5.4)$$

where the drift term reverts the trajectories of γ towards their long term mean θ at the adjustment speed k , while δ describes the volatility. This model can be extended to a *multifactor* version:

The model follows from the assumption of n independent factors that influence the interest rate r and default intensity γ . They are represented as linear combinations with positive weights on the factors:

$$r_t = \sum_{i=1}^n w_i x_i(t) \quad (5.5)$$

and

$$\gamma_t = \sum_{i=1}^n \hat{w}_i x_i(t), \quad (5.6)$$

where factor x_i follows the CIR model

$$dx_i(t) = (\alpha_i - \beta_i x_i(t))dt + \delta_i \sqrt{x_i(t)} dW_i(t). \quad (5.7)$$

The W_i 's are assumed to be mutually independent. For nonnegative W_i one can only produce positive correlation between r_t and γ_t . However, in order to generate negative correlation, one can introduce the factors y_i following

$$dy_i(t) = (\alpha_i - \beta_i y_i(t))dt - \delta_i \sqrt{y_i(t)} dW_i(t).$$

The $m < n$ first factors influence the interest rate (its weights for $i \geq m$ are thus zero), while all n factors will describe the dynamics of the default intensity.

ZCB Pricing

In [8] it is shown that for $c > 0$,

$$\mathbb{E}[e^{-\int_t^T cx_i(s)ds}|\mathcal{F}_t] = H_{1i}(T-t, c)e^{-H_{2i}(T-t, c)cx_i(t)}, \quad (5.8)$$

where

$$H_{1i}(T-t, c) := \left\{ \frac{2\gamma_i e^{\frac{1}{2}(\gamma_i + \beta_i)(T-t)}}{(\gamma_i + \beta_i)(e^{\gamma_i(T-t)} - 1) + 2\gamma_i} \right\}^{2\frac{\alpha_i}{\delta_i}}, \quad (5.9)$$

$$H_{2i}(T-t, c) := \frac{2(e^{\gamma_i(T-t)} - 1)}{(\gamma_i + \beta_i)(e^{\gamma_i(T-t)} - 1) + 2\gamma_i} \quad (5.10)$$

and

$$\gamma_i := \sqrt{\beta_i^2 + 2c\delta_i^2}. \quad (5.11)$$

Substitution of (5.5) in (5.8) gives (by independence of the factors) the default-free ZCB price

$$\begin{aligned} B(t, T) &= \mathbb{E}[e^{-\int_t^T r_s ds}|\mathcal{F}_t] = \mathbb{E}[e^{-\sum_{i=1}^n \int_t^T w_i x_i(s) ds}|\mathcal{F}_t] = \\ &= \prod_{i=1}^n \mathbb{E}[e^{-\int_t^T w_i x_i(s) ds}|\mathcal{F}_t] = \prod_{i=1}^n H_{1i}(T-t, w_i)e^{-H_{2i}(T-t, w_i)w_i x_i(t)}. \end{aligned} \quad (5.12)$$

The defaultable ZCB price follows by the same argument, now with $c = w_i + \bar{w}_i$:

$$B^d(t, T) = \mathbb{E}[e^{-\int_t^T \sum_{i=1}^n (w_i + \bar{w}_i)x_i(s) ds}|\mathcal{F}_t] = \prod_{i=1}^n H_{1i}(T-t, w_i + \bar{w}_i)e^{-H_{2i}(T-t, w_i + \bar{w}_i)(w_i + \bar{w}_i)x_i(t)}. \quad (5.13)$$

Pricing with recovery

As discussed in chapter 2,

$$B_\delta^d(t, T) = \mathbb{E}[\delta e^{-\int_t^T r_s ds}|\mathcal{F}_t] + \mathbb{E}[(1-\delta)e^{-\int_t^T r_s + \lambda_s ds}|\mathcal{F}_t]$$

for a defaultable ZCB with recovery. The assumption of a deterministic recovery simplifies the price to a linear combination of the default-free and defaultable price given as derived above:

$$B_\delta^d(t, T) = \delta B(t, T) + (1-\delta)B^d(t, T). \quad (5.14)$$

5.3 The Vasicek Model and ZCB Pricing

Let the dynamics of the default intensity be given by the Vasicek model, i.e.

$$d\gamma_t = k(\theta - \gamma_t)dt + \sigma dW_t, \quad (5.15)$$

where k , θ and σ are constants. k describes the reversion rate, i.e. the rate of which the trajectories of γ_t will regroup around their long term mean θ . σ denotes the volatility of the default intensity, whereas high values of σ indicate a large degree of randomness in the model. The main drawback of the Vasicek model is the fact that its trajectories may admit negative values (as opposed to the CIR model). However, it turns out to be a good model for analytical purposes, and is reasonably simple to implement by e.g. Euler discretization.

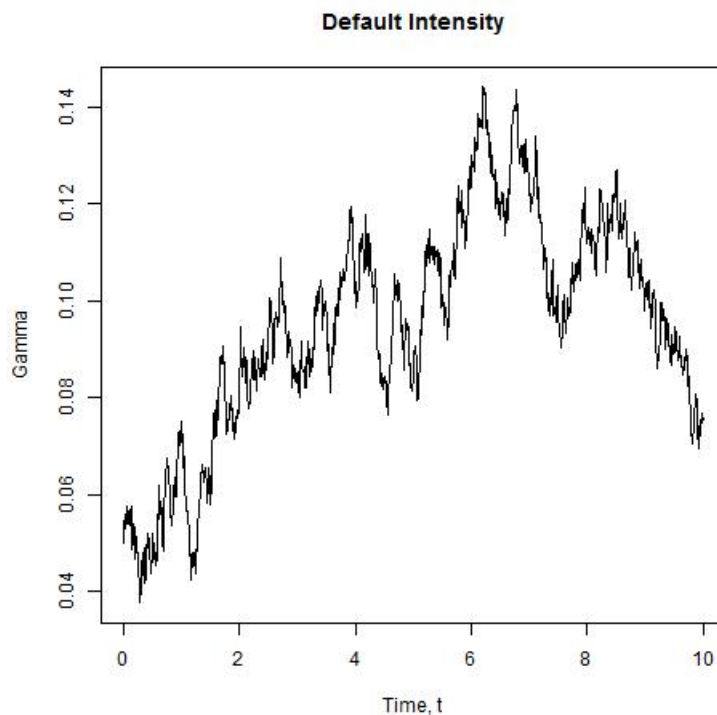


Figure 3.1: A simulated path of the Vasicek model with parameter values $\gamma_0 = 0.05$, $k = 0.3$, $\theta = 0.1$ and $\sigma = 0.03$.

The solution of (5.15) is found by applying Theorem (1.2.1) (Itô's formula) (See the appendix):

$$\gamma_t = \gamma_0 e^{-kt} + \theta(1 - e^{-kt}) + \sigma \int_0^t e^{k(s-t)} dW_s, \quad (5.16)$$

which follows the Gaussian distribution with mean

$$\mu_t := \gamma_0 e^{-kt} + \theta(1 - e^{-kt}) \quad (5.17)$$

and variance

$$V_t := \mathbb{E}[\gamma_t^2] = \sigma^2 e^{-2kt} \mathbb{E}[(\int_0^t e^{ks} dW_s)^2].$$

By Proposition (1.2.1) (Itô's isometry), the variance becomes

$$V_t = \sigma^2 e^{-2kt} \mathbb{E}[\int_0^t |e^{2ks}| ds] = \frac{\sigma^2}{2k} (1 - e^{-2kt}). \quad (5.18)$$

5.3.1 Default-free ZCB Pricing

We now want to derive an expression for the default-free ZCB price $B(t, T)$. Assume the interest rate r_t follows the Vasicek model, i.e.

$$dr_t = k(\theta - r_t)dt + \sigma dW_t. \quad (5.19)$$

Since r_t is Gaussian, so is the process $X_t := \int_0^t r_s ds$ (with a certain mean a and variance b^2). We can thus utilize the property that $\mathbb{E}[e^{-X_t}] = e^{-a + \frac{1}{2}b^2}$. We begin by finding an expression for X :

$$\begin{aligned} X_t &:= \int_0^t r_s ds = \int_0^t (\gamma_0 e^{-ks} + \theta(1 - e^{-ks}) + \sigma \int_0^s e^{k(s-u)} dW_u) ds = \\ &\quad \frac{\gamma_0}{k} (1 - e^{-kt}) + \frac{\theta}{k} (tk + e^{-kt} - 1) + \sigma \int_0^t (\int_0^s e^{k(s-u)} dW_u) ds. \end{aligned}$$

As Itô integrals have zero expectation, the mean of X becomes

$$a = \frac{\gamma_0}{k} (1 - e^{-kt}) + \frac{\theta}{k} (tk + e^{-kt} - 1). \quad (5.20)$$

In [10] it is shown that

$$\text{cov}(X_v, X_w) = \frac{\sigma^2}{2k} e^{-k(v+w)} (e^{2k(v \wedge w)} - 1), \quad (5.21)$$

which gives the variance

$$b^2 = \mathbb{V}(\int_0^t X_v dv) = \text{cov}(\int_0^t X_v dv, \int_0^t X_w dw) = \int_0^t \int_0^t \text{cov}(X_v, X_w) dv dw$$

$$= \int_0^t \int_0^t \frac{\sigma^2}{2k} e^{-k(v+w)} (e^{2k(v \wedge w)} - 1) dv dw = \frac{\sigma^2}{2k^3} (2kt - 3 + 4e^{-kt} - e^{-2kt}). \quad (5.22)$$

Now,

$$\mathbb{E}[-\int_t^T r_s ds] = -\frac{\gamma_0}{k} (1 - e^{-k(T-t)}) - \frac{\theta}{k} ((T-t)k + e^{-k(T-t)} - 1), \quad (5.23)$$

while

$$\mathbb{V}[-\int_t^T r_s ds] = \frac{\sigma^2}{2k^3} (2k(T-t) - 3 + 4e^{-k(T-t)} - e^{-2k(T-t)}). \quad (5.24)$$

Furthermore, r_t is a Markov process (see e.g. Karatzas and Shreve, p. 355), one can thus represent the default-free ZCB price as

$$B(t, T, r_t) = \mathbb{E}[e^{-\int_t^T r_s ds} | \mathcal{F}_t] = \mathbb{E}[e^{-\int_t^T r_s ds} | r_t] = \mathbb{E}[e^{-\int_t^T r_s ds}], \quad (5.25)$$

which again by (5.23) and (5.24) becomes

$$\begin{aligned} B(t, T, r_t) &= \exp\{\mathbb{E}[-\int_t^T r_s ds] + \frac{1}{2}\mathbb{V}[-\int_t^T r_s ds]\} = \\ &= \exp\left\{-\frac{(r_t - \theta)(1 - e^{-k(T-t)})}{k} - \theta(T-t) + \frac{\sigma^2}{4k^3} (2k(T-t) - 3 + 4e^{-k(T-t)} - e^{-2k(T-t)})\right\} \\ &= \exp\{-A(t, T)r_t + C(t, T)\}, \end{aligned} \quad (5.26)$$

where

$$A(t, T) := \frac{1 - e^{-k(T-t)}}{k} \quad (5.27)$$

and

$$C(t, T) := (\theta - \frac{\sigma^2}{2k})(A(t, T) - (T-t)) - \frac{\sigma^2}{4k} A^2(t, T). \quad (5.28)$$

5.3.2 Defaultable ZCB Pricing with Correlated Default Intensity ♠

Let

$$dr_t = k(\theta - r_t)dt + \sigma dW_t^1$$

and

$$d\gamma_t = k(u - \gamma_t)dt + \tau(\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2) \quad (5.29)$$

describe the default intensity γ_t and interest rate r_t , where W_t^1 and W_t^2 are assumed to be independent Brownian motions. With this setup, γ_t and r_t will be dependent random variables with correlation $\rho \in [-1, 1]$. The goal now is to calculate the defaultable bond price $B^d(t, T) = \mathbb{E}[e^{-\int_t^T \gamma_s + r_s ds} | \mathcal{F}_t]$. $\gamma_t + r_t$ will be a Markov process as long as one picks the same k for both equations. We have that

$$r_t = e^{-kt} \gamma_0 + \theta(1 - e^{-kt}) + \sigma \int_0^t e^{-k(t-s)} dW_s^1$$

and

$$\gamma_t = e^{-kt} r_0 + u(1 - e^{-kt}) + \tau \int_0^t \rho e^{-k(t-s)} dW_s^1 + \tau \int_0^t \sqrt{1 - \rho^2} e^{-k(t-s)} dW_s^2. \quad (5.30)$$

For simplicity, denote

$$\int_0^T r_t dt = I_1^r + I_2^r + I_3^r \quad (5.31)$$

and

$$\int_0^T \gamma_t dt = I_1^\gamma + I_2^\gamma + I_3^\gamma + I_4^\gamma, \quad (5.32)$$

where I_3^r, I_3^γ and I_4^γ are the three stochastic integrals. The expectation, a , follows directly from

$$\begin{aligned} a &= \mathbb{E}[I_1^\gamma + I_2^\gamma + I_3^\gamma + I_1^r + I_2^r + I_3^r] = I_1^\gamma + I_2^\gamma + I_1^r + I_2^r = \\ &\quad \frac{r_0 + \gamma_0}{k}(1 - e^{-kT}) + \frac{u + \theta}{k}(kT + e^{-kT} - 1), \end{aligned}$$

as stochastic integrals are martingales and thus have expectation equal to zero. The variance, b^2 , is found by

$$b^2 = \mathbb{V}(\int_0^T (\gamma_s + r_s) ds) = \mathbb{V}(\int_0^T \gamma_s ds) + \mathbb{V}(\int_0^T r_s ds) + 2\text{cov}(\int_0^T \gamma_s ds, \int_0^T r_s ds). \quad (5.33)$$

One can show that

$$\text{cov}(\int_0^T \gamma_s ds, \int_0^T r_s ds) = \mathbb{E}[I_3^r(I_3^\gamma + I_4^\gamma)], \quad (5.34)$$

which by independence between W_t^1 and W_t^2 gives

$$\text{cov}(\int_0^T \gamma_s ds, \int_0^T r_s ds) = \frac{\sigma \tau \rho}{2k}(1 - e^{-2kT}). \quad (5.35)$$

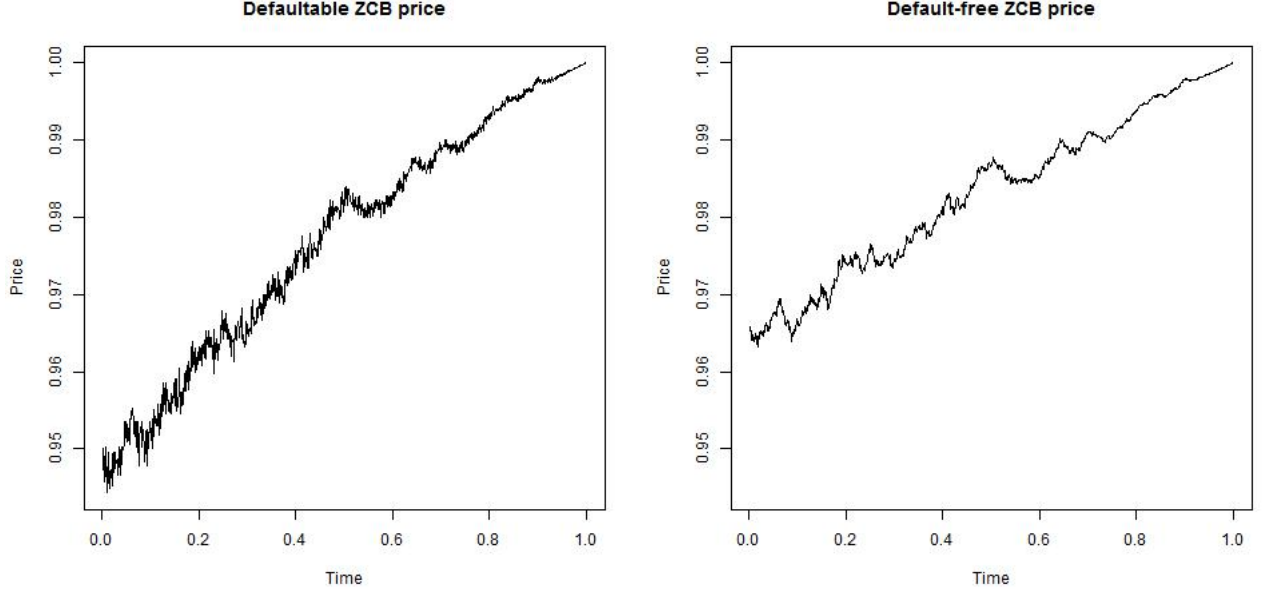


Figure 3.2: *Left: Defaultable bond price with interest rate parameters $r_0 = 0.03$, $k = 0.2$, $\theta = 0.1$, $\sigma = 0.02$ and default risk parameters $\gamma_0 = 0.03$, $k = 0.2$, $\tau = 0.03$, $u = 0.2$, $\rho = 0.2$. Right: Default-free bond price.*

By similar calculations,

$$\mathbb{V}\left(\int_0^T \gamma_t dt\right) = \frac{\tau^2}{k^2}\left(T + \frac{4e^{-kT} - e^{-2kT} - 3}{2k}\right) \quad (5.36)$$

and

$$\mathbb{V}\left(\int_0^T r_t dt\right) = \frac{\sigma^2}{k^2}\left(T + \frac{4e^{-kT} - e^{-2kT} - 3}{2k}\right), \quad (5.37)$$

hence

$$b^2 = \frac{(\sigma^2 + \tau^2)}{k^2}\left(T + \frac{4e^{-kT} - e^{-2kT} - 3}{2k}\right) + \frac{\sigma\tau\rho}{k}(1 - e^{-2kT}). \quad (5.38)$$

By replacing γ_0 and r_0 by γ_t and r_t , and T by $T-t$ (due to the Markov property of $\gamma_t + r_t$), the final expression for the defaultable ZCB price at all time points $t \in [0, T]$ is given by:

$$B^d(t, T) = \mathbb{E}\left[e^{-\int_t^T \gamma_s + r_s ds} \middle| \mathcal{F}_t\right] = e^{\frac{1}{2}b^2(t, T) - a(t, T)}, \quad (5.39)$$

where

$$a(t, T) := \frac{\gamma_t + r_t}{k}(1 - e^{-k(T-t)}) + \frac{u + \theta}{k}(k(T-t) + e^{-k(T-t)} - 1) \quad (5.40)$$

and

$$b^2(t, T) := \frac{(\sigma^2 + \tau^2)}{k^2} \left[(T - t) + \frac{4e^{-k(T-t)} - e^{-2k(T-t)} - 3}{2k} \right] + \frac{\sigma\tau\rho}{k} (1 - e^{-2k(T-t)}). \quad (5.41)$$

□

Chapter 6

Lévy Processes

The most basic examples of Lévy processes are Brownian motion and compound Poisson processes. The latter are Poisson processes with random jump sizes. In fact, every Lévy process can be decomposed into a sum of a Brownian motion with drift and (perhaps infinitely many) centered compound Poisson processes. This is called the Lévy-Itô decomposition. This chapter is based on material from [5], [11] and [7]. We begin by introducing the basic concepts on Lévy processes, such as infinite divisibility, characteristic triplets and other distributional properties. Specific Lévy processes will be discussed in detail, additionally in the setting of acting as driving processes for the Vasicek model. Simulation techniques and pricing methods are then briefly explained in the case of such a model. Finally, Lévy processes as α -stable and Variance Gamma processes will be calibrated to market data in order to price defaultable bonds, taking into account jumps in the default intensity, described by a Lévy driven Vasicek model.

6.1 Distributional Properties of Lévy Processes

We start by defining the Lévy process:

Definition 6.1.1 (Lévy process). *A cadlag¹ stochastic process $\{L_t\}_{t \geq 0} \in \mathbb{R}^d$ with $L_0 = 0$ is called Lévy if it satisfies the following three properties:*

1. *Independence: For the time points $t_0 < t_1 < \dots < t_n$, the increments $L_{t_0}, L_{t_1} - L_{t_0}, \dots, L_{t_n} - L_{t_{n-1}}$ are independent random variables.*
2. *Stationarity: The distribution of $L_{t+h} - L_t$ is independent of t .*

¹Cadlag refers to the property of being right-continuous and having left limits.

3. *Stochastic continuity: For any t , the probability of seeing a jump is zero.*

The last condition ensures the exclusion of jumps at fixed times, any jump of a Lévy process must thus arrive at random times. A notable property is the fact that the sum of any Lévy process sampled at fixed time intervals gives a random walk:

Definition 6.1.2 (Infinite divisibility). *Let P be a probability distribution on \mathbb{R}^d . If there exists n i.i.d. random variables X_1, \dots, X_n such that the sum $X_1 + \dots + X_n \stackrel{d}{=} P$, then P is called an infinitely divisible probability distribution.*

It turns out, that if L_t is a Lévy process, it has an infinitely divisible distribution for any $t > 0$. It can be represented as a sum of its n i.i.d. increments,

$$L_t = L_0 + L_{\frac{t}{n}} - L_0 + \dots + L_t - L_{\frac{t(n-1)}{n}} = \sum_{k=1}^n \Delta L_{t_k},$$

where each increment ΔL_{t_k} follows the same distribution as $L_{\frac{t}{n}}$. Conversely, it can be shown that for any infinitely divisible distribution there exists a Lévy process X_t which at $t = 1$ follows the same distribution.

Definition 6.1.3 (Characteristic function). *The characteristic function $\Phi_L(z)$ of a stochastic process $\{L_t\}_{t \geq 0} \in \mathbb{R}^d$ is given by*

$$\Phi_L(z) = \mathbb{E}[e^{iz \cdot L_t}] = e^{t\psi(z)}, \quad z \in \mathbb{R}^d. \quad (6.1)$$

The function ψ is called the characteristic exponent of L_t .

Characteristic functions are related to the n -th moments of L and determines distributional properties of stochastic processes. The last equality of (6.1) follows from the following observations: Let $\Phi_z(t) = \Phi_{L_t}(z)$. Then

$$\begin{aligned} \Phi_z(t+h) &= \mathbb{E}[e^{iz \cdot L_{t+h}}] = \mathbb{E}[e^{iz \cdot (L_{t+h} - L_h)} e^{iz \cdot L_h}] = \\ &= \mathbb{E}[e^{iz \cdot (L_{t+h} - L_h)}] \mathbb{E}[e^{iz \cdot L_h}] = \Phi_z(t) \Phi_z(h), \end{aligned} \quad (6.2)$$

where the third equality follows from the third property of Definition 6.1.1. (6.2) has the unique solution

$$\Phi_z(t) = e^{t\psi(z)}, \quad z \in \mathbb{R}^d. \quad (6.3)$$

Finally, the Lévy process L_1 follows some infinitely divisible distribution.

The jump at $t \geq 0$ of a Lévy process is given by

$$\Delta L_t = L_t - L_{t-}, \quad (6.4)$$

which motivates the notion of a Lévy measure:

Definition 6.1.4 (Lévy measure [5]). *Let $\{L_t\}_{t \geq 0}$ be a Lévy process on \mathbb{R}^d . The measure ν on \mathbb{R}^d defined by:*

$$\nu(A) = \mathbb{E}[\#\{t \in [0, 1] : \Delta L_t \neq 0, \Delta L_t \in A\}], \quad A \in \mathcal{B}(\mathbb{R}^d) \quad (6.5)$$

is called the Lévy measure on X : $\nu(A)$ is the expected number, per unit time, of jumps whose size belongs to A .

Poisson processes and martingales

Poisson processes are one of the simplest examples of Lévy processes and their properties will be utilized in the following theory. We start by giving its definition:

Definition 6.1.5 (Poisson process). *An \mathcal{F} -adapted cadlag stochastic process $\{L_t^P\}_{t \geq 0}$ on $[0, T]$ is called a Poisson process with intensity $\lambda > 0$, if it satisfies the following properties:*

1. $L_0^P = 0$.
2. For any $0 \leq t \leq s \leq T$, the increment $L_s^P - L_t^P$ is independent of the filtration \mathcal{F}_t .
3. For any $0 \leq t \leq s \leq T$, the increment $L_s^P - L_t^P$ follows the Poisson distribution with parameter $\lambda(s - t)$.

Compensated Poisson processes are important building blocks of Lévy processes. They are given by the following definition:

Definition 6.1.6. *Let L_t^P be a Poisson process with intensity $\lambda > 0$. The compensated Poisson process, \bar{L}_t^P , is defined by*

$$\bar{L}_t^P = L_t^P - \lambda t. \quad (6.6)$$

It can be shown that every compensated Poisson process is a martingale:

Proposition 6.1.1 (Martingale property of compensated Poisson processes). *Let $L_t^{\bar{C}P}$ be a compensated Poisson process. Then $L_t^{\bar{C}P}$ is a martingale.*

Proof

We check the properties of (1.2.3):

1. \bar{L}_t^P is adapted, as L_t^P is adapted by definition (see Appendix for definition)
2. $\mathbb{E}[|\bar{L}_t^P|] < \infty$, as $\mathbb{E}[|L_t^P|] < \infty$ for every $t \in [0, T]$.

3. For every $0 \leq t \leq s \leq T$,

$$\mathbb{E}[\bar{L}_s^P | \mathcal{F}_t] = \mathbb{E}[L_s^P - \lambda s | \mathcal{F}_t] = \mathbb{E}[L_s^P - L_t^P + L_t^P - \lambda(s - t + t) | \mathcal{F}_t] = L_t^P - \lambda t = \bar{L}_t^P.$$

□

Compound Poisson process

A compound Poisson process is a stochastic process L_t^{CP} with intensity $\lambda > 0$ whose jump sizes X are i.i.d. and follow a distribution d . It is defined by

$$L_t^{CP} := \sum_{i=1}^{L_t^P} X_i. \quad (6.7)$$

Here, L_t^P is an ordinary Poisson process with intensity $\lambda > 0$, being independent of X . Note that for $X_i \equiv 1$, L_t^P is simply a Poisson process. Let $J_{L^{CP}}(A) = \#\{(t, \Delta L_t^{CP}) \in A\}$ count the number of jumps of L_t^{CP} on $[0, t]$ with size belonging to the measurable set $A \subset \mathbb{R}^d$. $J_{L^{CP}}$ is referred to as a *jump measure*, and is in fact a Poisson random measure with intensity measure $\nu(dx)dt$. Due to this fact, every compound Poisson process can be represented as

$$L_t^{CP} = \sum_{s \in [0, t]} \Delta L_s = \int_{[0, t] \times \mathbb{R}^d} x J_L(ds \times dx). \quad (6.8)$$

This sum will converge, as compound Poisson processes admit finitely many jumps on any interval $[0, t]$. This is a special case of a more general case where every Lévy process can be decomposed in a similar way, utilizing compound Poisson processes:

Proposition 6.1.2 (Lévy-Itô decomposition [5]). *Let $\{L_t\}_{t \geq 0}$ be a Lévy process on \mathbb{R}^d and ν its Lévy measure, given by Definition 6.1.4. Then,*

- ν is a Radon measure on $\mathbb{R}^d \setminus \{0\}$ and verifies:

$$\int_{|x| \leq 1} |x|^2 \nu(dx) < \infty \quad , \quad \int_{|x| \geq 1} \nu(dx) < \infty. \quad (6.9)$$

- The jump measure of L , denoted by J_L , is a Poisson random measure on $[0, \infty) \times \mathbb{R}^d$ with intensity measure $\nu(dx)dt$.
- There exists a vector γ and d -dimensional Brownian motion $\{B_t\}_{t \geq 0}$ with covariance matrix A such that

$$L_t = \gamma t + B_t + L_t^l + \lim_{\epsilon \downarrow 0} \hat{L}_t^\epsilon, \quad (6.10)$$

where

$$L_t^l = \int_{|x| \geq 1, s \in [0, t]} x J_L(ds \times dx) \quad (6.11)$$

and

$$\hat{L}_t^\epsilon = \int_{\epsilon \leq |x| < 1, s \in [0, t]} x \{J_L(ds \times dx) - \nu(dx)ds\} \equiv \int_{\epsilon \leq |x| < 1, s \in [0, t]} x \hat{J}_L(ds \times dx). \quad (6.12)$$

The terms in (6.10) are independent and the convergence in the last term is almost sure and uniform in t on $[0, T]$.

The Lévy-Itô decomposition approximates any Lévy process by arbitrary precision as a sum of a Brownian motion with drift and a sum of (maybe infinitely many) compensated compound Poisson processes. The first two terms consist of a Brownian motion with drift γt . The next term, L_t^l , is a compound Poisson process that takes care of the jumps that are larger than one in size. Since every Lévy process is cadlag, there are finitely many jumps of this size (and thus no convergence problems). The last term, \hat{L}_t^ϵ takes care of the jumps that are smaller than one. The reason why we cannot let ϵ be zero directly, is because the Lévy measure blows up at zero if there are infinitely many jumps whose sizes are smaller than one. Compensated (centered) compound processes are used instead for ensuring convergence. The triple (A, ν, γ) is called the *characteristic triplet* of the Lévy process. It characterizes the unique distribution of the Lévy process through its characteristic function:

Theorem 6.1.1 (Lévy-Khinchin representation [5]). *Let $\{L_t\}_{t \geq 0} \in \mathbb{R}^d$ be a Lévy process with characteristic triplet (A, ν, γ) . Then*

$$\mathbb{E}[e^{iz \cdot L_t}] = e^{t\psi(z)}, \quad z \in \mathbb{R}^d$$

with

$$\psi(z) = -\frac{1}{2}z \cdot Az + i\gamma \cdot z + \int_{\mathbb{R}^d} (e^{iz \cdot x} - 1 - iz \cdot x \mathbb{1}_{|x| \leq 1}) \nu(dx).$$

Remark: Since any infinitely divisible distribution is the distribution of some Lévy process at $t = 1$, they can also be represented in a similar way.

6.2 The Lévy-driven Vasicek Model and its Distribution

Consider the Vasicek model, driven by a Lévy process L_t :

$$d\gamma_t = k(\theta - \gamma_t)dt + \sigma dL_t. \quad (6.13)$$

As every Lévy process is a semimartingale, so is the solution γ_t of (6.13). It is obtained by applying Itô's formula for semimartingales:

Proposition 6.2.1. [5] *Let γ_t be a semimartingale. For any $C^{1,2}$ -function $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$,*

$$\begin{aligned} f(t, \gamma_t) - f(0, \gamma_0) &= \int_0^t \frac{\partial f}{\partial s}(s, \gamma_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, \gamma_{s-}) d\gamma_s \\ &+ \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, \gamma_{s-}) d[\gamma, \gamma]_s^c + \sum_{0 \leq s \leq t, \Delta X_s \neq 0} [f(s, \gamma_s) - f(s, \gamma_{s-}) - \Delta \gamma_s \frac{\partial f}{\partial x}(s, \gamma_{s-})] \end{aligned}$$

In similarity to the Gaussian case, the integrating factor $f(t, \gamma_t) = e^{kt} \gamma_t$ solves (6.13). We can disregard the third term on the RHS with continuous quadratic variation, as $\frac{\partial^2 f}{\partial x^2}(s, \gamma_{s-}) = 0$. The sum also vanishes due to this particular choice of f . Thus,

$$\gamma_t = e^{-kt} \gamma_0 + \theta(1 - e^{-kt}) + \sigma \int_0^t e^{k(s-t)} dL_s. \quad (6.14)$$

The details can be found in the appendix. The distributional properties of γ_t are given in terms of its characteristic triplet, which can be expressed in terms of the characteristic triplet (A, ν, δ) of L_t . The characteristic function of γ_t becomes

$$e^{\Psi_\gamma(t)} = \mathbb{E}[e^{iu\gamma_t}] = \mathbb{E}[e^{iu(\gamma_0 e^{-kt} + \theta(1 - e^{-kt})) + \int_0^t \sigma e^{k(s-t)} dL_s}]. \quad (6.15)$$

Further calculations are based on the following result:

Lemma 6.2.1. [5] *Let $f : [0, T] \rightarrow \mathbb{R}$ be left-continuous and $(L_t)_{t \geq 0}$ be a Lévy process. Then*

$$\mathbb{E}[e^{i \int_0^T f(t) dL_t}] = e^{\int_0^T \psi(f(t)) dt}$$

where $\psi(u)$ is the characteristic exponent of L .

Applying Lemma (6.2.1) yields

$$e^{\Psi_\gamma(t)} = e^{iu(\gamma_0 e^{-kt} + \theta(1 - e^{-kt})) + \int_0^t \Psi(\sigma e^{k(s-t)}) ds}, \quad (6.16)$$

where Ψ is the characteristic function of L_t :

$$\Psi(u) = -\frac{1}{2}Au^2 + i\delta u + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux\mathbb{1}\{|x| \leq 1\})\nu(dx). \quad (6.17)$$

Integration of

$$\begin{aligned} \Psi(u\sigma e^{k(s-t)}) &= -\frac{1}{2}Au^2\sigma^2 e^{2k(s-t)} + i\delta u\sigma e^{k(s-t)} + \\ &\int_{-\infty}^{\infty} (e^{iux\sigma e^{k(s-t)}} - 1 - iux\sigma e^{k(s-t)}\mathbb{1}\{|x| \leq 1\})\nu(dx) \end{aligned}$$

yields

$$\begin{aligned} \int_0^t \Psi(u\sigma e^{k(s-t)})ds &= \int_0^t \left(-\frac{1}{2}Au^2\sigma^2 e^{2k(s-t)} + i\delta u\sigma e^{k(s-t)}\right)ds \\ &+ \int_0^t \int_{-\infty}^{\infty} (e^{iux\sigma e^{k(s-t)}} - 1 - iux\sigma e^{k(s-t)}\mathbb{1}\{|x| \leq 1\})\nu(dx)ds = I_1 + I_2 + I_3, \end{aligned}$$

where

$$I_1 := \int_0^t -\frac{1}{2}Au^2\sigma^2 e^{2k(s-t)}ds = \frac{Au^2\sigma^2}{4k}(e^{-2kt} - 1)$$

and

$$I_2 := \int_0^t i\delta u\sigma e^{k(s-t)}ds = \frac{i\delta u\sigma}{k}(1 - e^{-kt}).$$

(6.16) is then given by

$$\begin{aligned} \Psi_\gamma(u) &= iu(\gamma_0 e^{-kt} + \theta(1 - e^{-kt})) + \frac{1}{4k}Au^2\sigma^2(e^{-2kt} - 1) + \frac{i\delta u\sigma}{k}(1 - e^{-kt}) + \\ &\int_0^t \int_{-\infty}^{\infty} (e^{iux\sigma e^{k(s-t)}} - 1 - iux\sigma e^{k(s-t)}\mathbb{1}\{|x| \leq 1\})\nu(dx)ds. \end{aligned} \quad (6.18)$$

The characteristic triplet $(A_t^\gamma, \delta_t^\gamma, \nu_t^\gamma)$ of γ_t consists thus of the elements

$$A_t^\gamma = \frac{A\sigma^2}{2k}(1 - e^{-2kt}), \quad (6.19)$$

$$\delta_t^\gamma = \gamma_0 e^{-kt} + \theta(1 - e^{-kt}) + \frac{\delta\sigma}{k}(1 - e^{-kt}), \quad (6.20)$$

and

$$\nu_t^\gamma = \int_1^{e^{kt}} \nu(\xi B) \frac{d\xi}{k\xi}, \quad \forall B \in \mathcal{B}(\mathbb{R}). \quad (6.21)$$

[5] verifies that ν_t^γ is in fact a Lévy measure, where also a suitable change of variables yields (6.21).

6.3 α -stable Lévy Processes

Definition 6.3.1 (Stable distribution [5]). *A random variable $X \in \mathbb{R}^d$ is said to have a stable distribution if for every $a > 0$ there exist $b(a) > 0$ and $c(a) \in \mathbb{R}^d$ such that*

$$\Phi_X(z)^a = \Phi_X(zb(a))e^{ic(a)z}, \quad \forall z \in \mathbb{R}^d.$$

It is said to have a strictly stable distribution if

$$\Phi_X(z)^a = \Phi_X(zb(a)), \quad \forall z \in \mathbb{R}^d.$$

An example of an α -stable distribution is the Gaussian distribution. It corresponds to $\alpha = 2$, $b(a) = a^{\frac{1}{\alpha}}$ and $c(a) = \gamma a^{\frac{1}{2}}(a^{\frac{1}{2}} - 1)$ where γ denotes its mean. Brownian motion is a stochastic process with a Gaussian distribution and is therefore a 2-stable process. In fact, all 2-stable processes are Gaussian. Any stable distribution is the distribution of a Lévy process at a given time:

Proposition 6.3.1 (Stable distributions and Lévy processes [5]). *A distribution on \mathbb{R}^d is α -stable with $0 < \alpha < 2$ if and only if it is infinitely divisible with characteristic triplet $(0, \nu, \gamma)$ and there exists a finite measure λ on S , a unit sphere of \mathbb{R}^d , such that*

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) \frac{dr}{r^{1+\alpha}}.$$

A distribution on \mathbb{R}^d is α -stable with $\alpha = 2$ if and only if it is Gaussian.

In the case of real-valued and one-dimensional α -stable distributions, for $0 < \alpha < 2$ the Lévy measure is on the form

$$\nu(x) = \frac{A}{x^{\alpha+1}} \mathbb{1}_{x>0} + \frac{B}{|x|^{\alpha+1}} \mathbb{1}_{x<0},$$

where A and B are positive constants. For the sake of simplicity, we will consider the *symmetric* α -stable distribution, i.e. $A = B$.

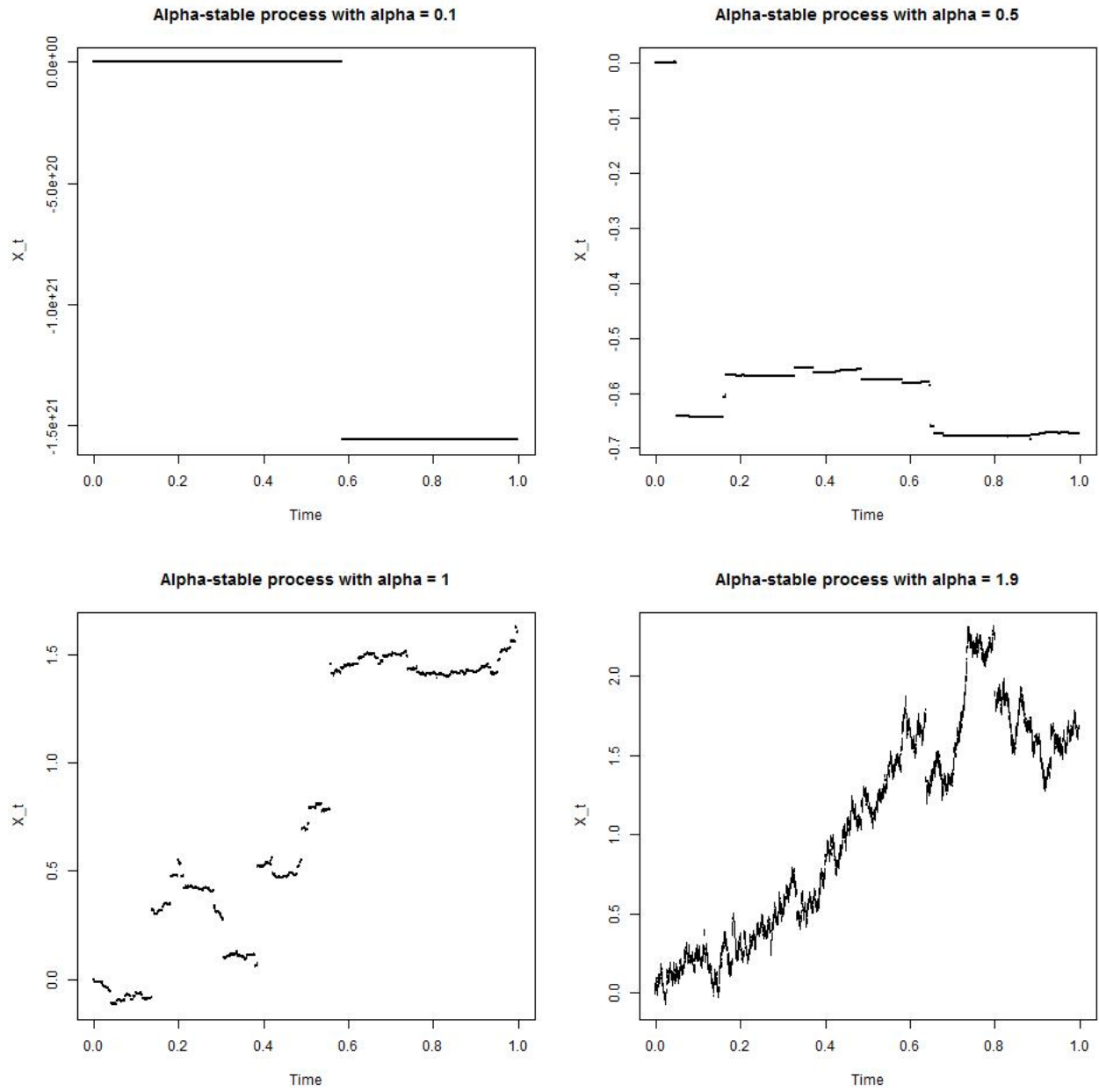


Figure 6.1: α -stable processes for $\alpha=0.1, 0.5, 1$ and 1.9 .

For low values of α , the distribution is heavy tailed and the sample paths consist mostly of large jumps. When α is near the value of 2, the distribution approaches the Gaussian distribution and one can see that the trajectories of the process seldom has large jumps,

but start resemble the continuous paths of Brownian motion. For alpha equal to one, the trajectories move by a mixture of small and large jumps.

Series representation [5]

If a series representation for $L_t = \sum_{i=1}^{\infty} A_i \mathbb{1}\{V_i < t\}$ is available,

$$\gamma_t = \gamma_0 e^{-kt} + \theta(1 - e^{-kt}) + \sum_{i=1}^{\infty} A_i e^{k(V_i - t)} \sigma \mathbb{1}\{V_i < t\}. \quad (6.22)$$

6.3.1 Simulation and Pricing with α -stable Lévy Processes

In general, there are three ways of constructing Lévy processes, each with their (dis)advantages. One is through the characteristic triplet of the Lévy process or by specifying an infinitely divisible distribution as the distribution of the Lévy process at $t = 1$. By utilizing the characteristic triplet, the pathwise properties are known, however, calibration and simulation is potentially difficult. In contrast to the second method, which makes simulation and estimation reasonably simple, but the pathwise properties may be unknown. The last method is done by time-changing a Brownian motion with an increasing Lévy process, which is reasonably simple to estimate. We will look at an example of this, more specifically the Variance Gamma process (see chapter 6.4).

The goal of this section is to find a discretized path of the symmetric α -stable process L_t^α . From [5] one has the following algorithm:

Algorithm 6.3.1 (Path of a symmetric α -stable process)

1. Draw n independent variables U_i from the uniform distribution on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
2. Draw n independent variables E_i from the standard exponential distribution.
3. The increments $\Delta L_{t_i}^\alpha$ are then given by

$$\Delta L_{t_i}^\alpha = (\Delta t_i)^{\frac{1}{\alpha}} \frac{\sin(\alpha U_i)}{(\cos(U_i))^{\frac{1}{\alpha}}} \left[\frac{\cos((1 - \alpha)U_i)}{E_i} \right]^{\frac{1 - \alpha}{\alpha}}, \quad (6.23)$$

where $\Delta t_i = t_i - t_{i-1}$ and $t_0 = 0$.

4. The trajectory of L_t^α is then given by the sum of the increments in (6.23):

$$L_{t_i}^\alpha = \sum_{k=1}^i \Delta L_{t_k}^\alpha. \quad (6.24)$$

The paths of the Vasicek model driven by a symmetric α -stable process can then be simulated by utilizing Algorithm 6.3.1.

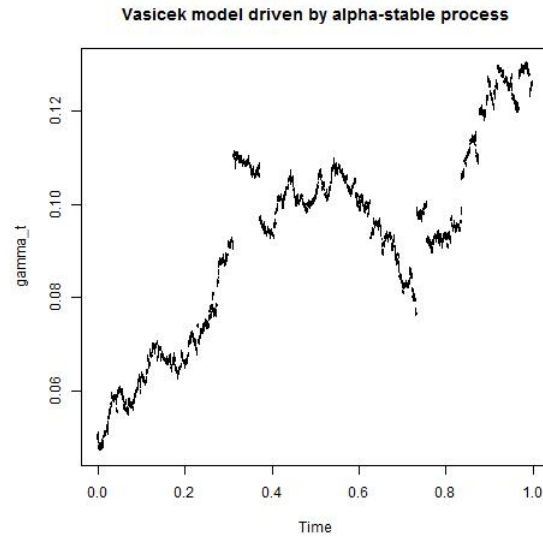


Figure 6.2: A sample path of the Vasicek model driven by an α -stable process with parameters $\alpha = 1.5$, $\gamma_0 = 0.05$, $k = 0.2$, $\theta = 0.15$ and $\sigma = 0.05$.

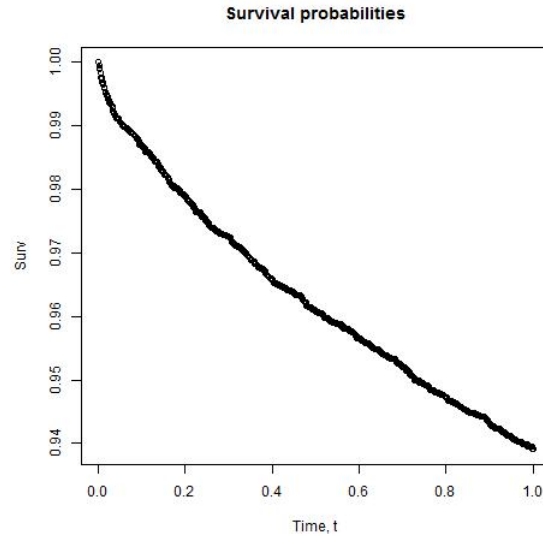


Figure 6.3: The corresponding survival probabilities to the default intensity given by the same parameters values as in Figure 6.2.

Bond pricing with α -stable processes

When the default intensity is given by (6.13), obtaining a closed form expression for the defaultable ZCB price

$$B^d(t, T) = \mathbb{E}_{\mathbb{Q}}[\exp(-\int_t^T (r_s + \gamma_s)ds)] \quad (6.25)$$

is difficult. However, by applying Algorithm 6.3.1. to simulate the differential dL_t , one can approximate the solution of (6.13) driven by an α -stable process by Euler discretization. The defaultable bond price (6.25) is then approximated by Monte Carlo integration.

6.3.2 Calibration to Market Data ♠

The goal is to find a probability distribution for the default times, consistent with the market prices. Let again $B(t, T)$ and $B^d(t, T)$ denote the ZCB prices of a defaultfree and defaultable bond, respectively. The survival probabilities are given by the ratio $P(t, T) = \frac{B^d(t, T)}{B(t, T)}$, in our setting of an intensity based model we thus obtain

$$P(t, T) = \frac{\mathbb{E}_{\mathbb{Q}}[\exp(-\int_t^T (r_s + \gamma_s)ds)|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}}[\exp(-\int_t^T r_s ds)|\mathcal{F}_t]}, \quad (6.26)$$

where the corresponding default probabilities are given by $Q(t, T) = 1 - P(t, T)$. Under the assumption of independence between γ_t and r_t , the survival probabilities simplify to

$$P(t, T) = \mathbb{E}_{\mathbb{Q}}[\exp(-\int_t^T \gamma_s ds)|\mathcal{F}_t], \quad (6.27)$$

as discussed in chapter 3.2. Given the historical interest rate data², it remains to model the default intensity γ_t . We attempt to model the credit spreads by means of an α -stable process with $\alpha = 2, 1.5, 1$ and 0.5 . The market data consists of yield curves of defaultable bonds between March 1999 and September 2008, taken from the Bank of Greece.³

Results

The case of $\alpha = 2$ corresponds to the Gaussian Vasicek model of the default intensity. When jumps are included (i.e. picking $\alpha = 0.5, 1$ and 1.5), the root mean squared error of the difference between the observed yield data and the theoretical yield decreases with decreasing values of α . Including heavy-tailed distributions (as α -stable processes with relatively small values of α) thus provides a better fit in this case, compared to the Gaussian

²<http://www.oenb.at/isaweb/report.do?report=10.4>

³http://www.bankofgreece.gr/Pages/en/Statistics/rates_markets/titloiieldimosiou/default.aspx

(which assigns very low probabilities to extreme events). The results are thus in line with the initial intuition.

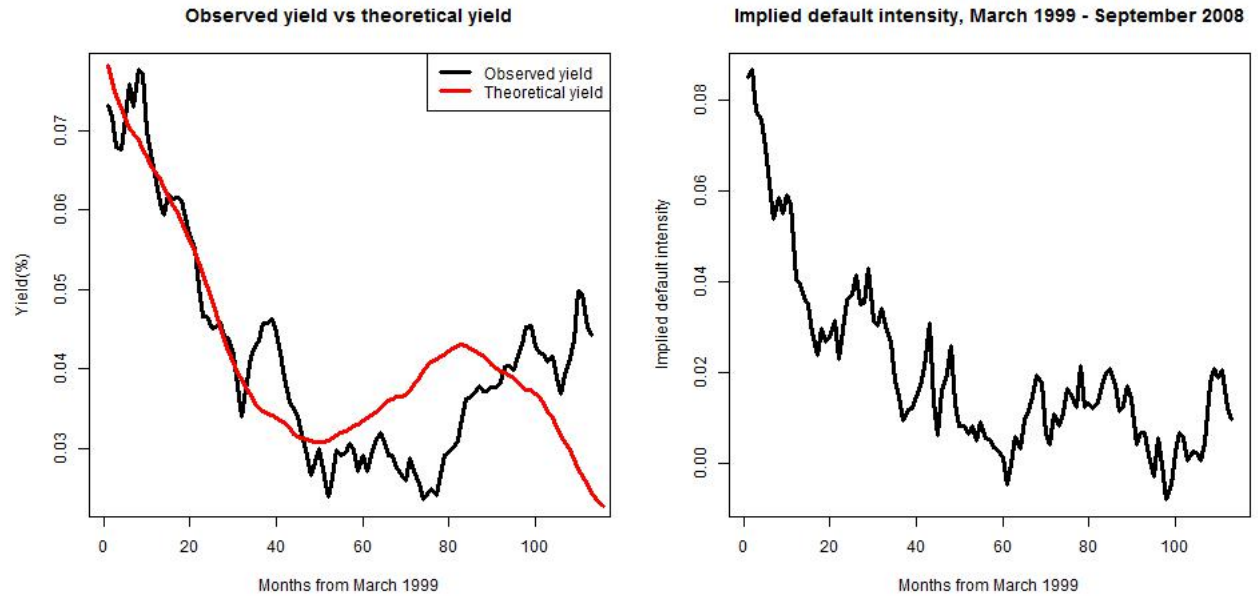


Figure 6.4: *Left: Yield data vs th. yield. Right: Implied default intensity for $\alpha = 2$.*

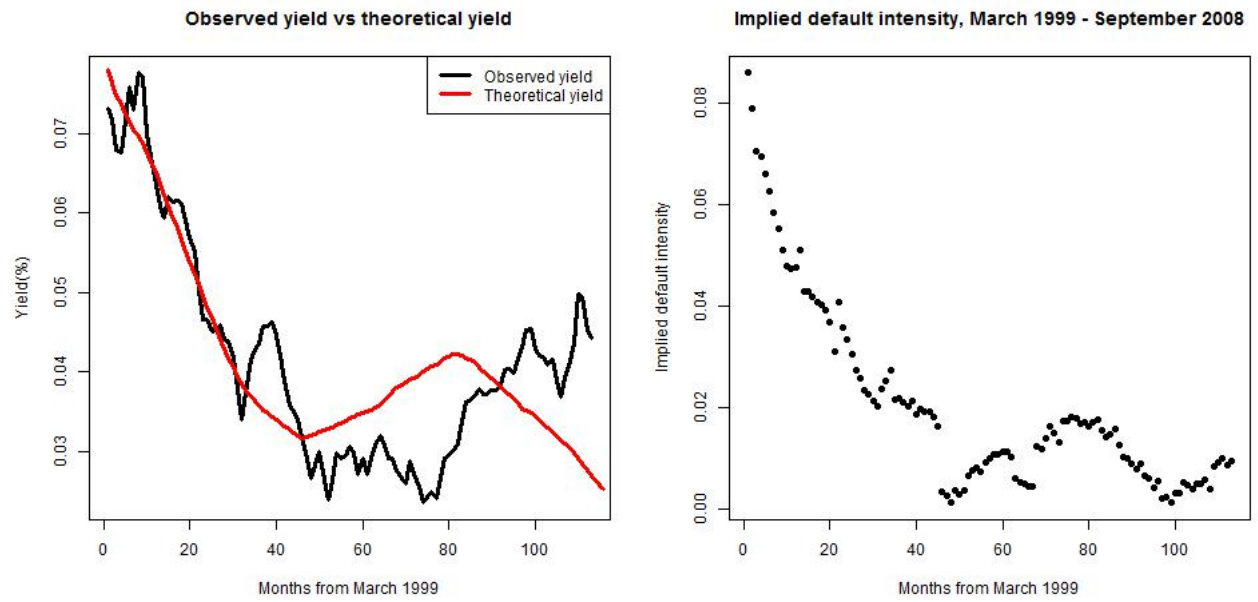


Figure 6.5: *Left: Yield data vs th. yield. Right: Implied default intensity for $\alpha = 1.5$.*

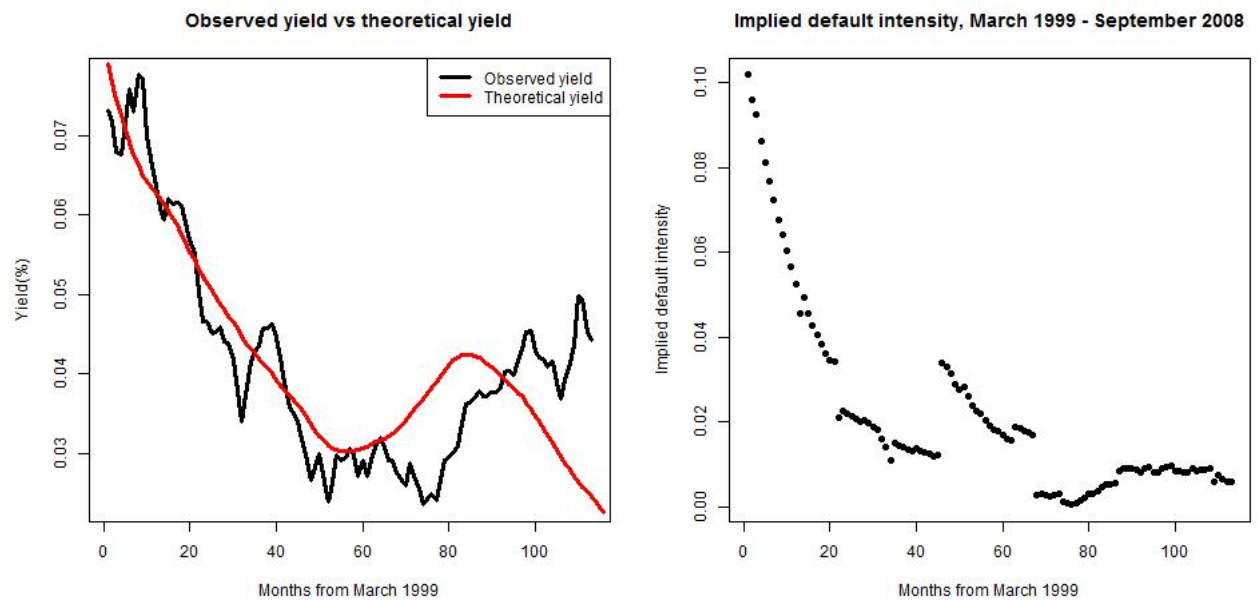


Figure 6.6: Left: Yield data vs th. yield. Right: Implied default intensity for $\alpha = 1$.

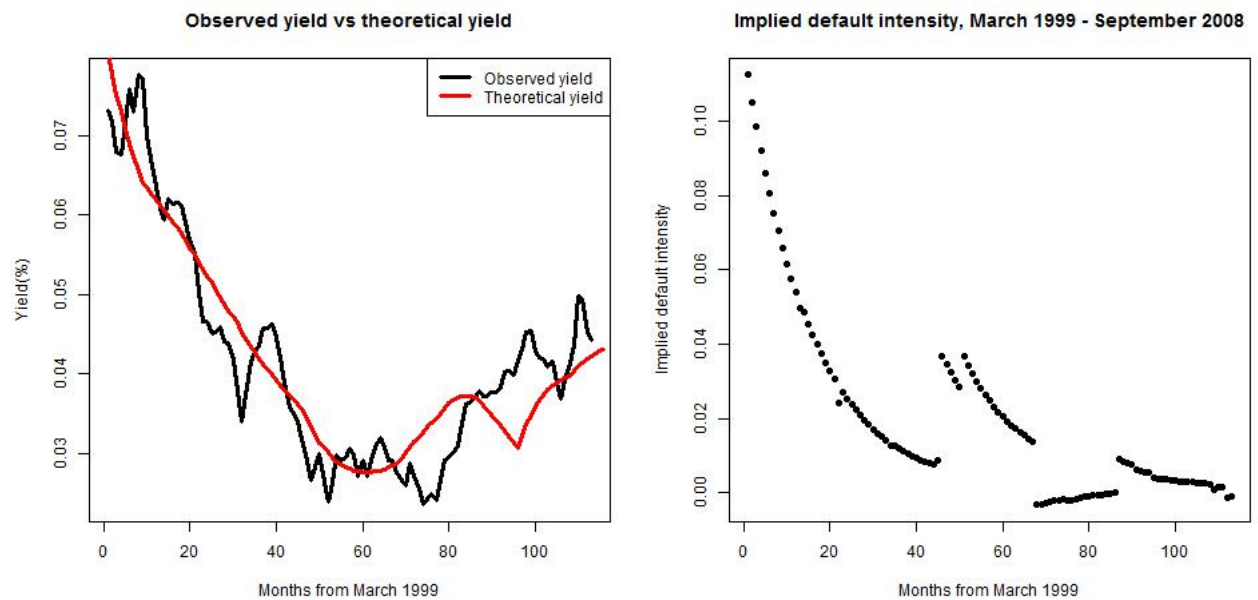


Figure 6.7: Left: Yield data vs th. yield. Right: Implied default intensity for $\alpha = 0.5$.

6.4 The Variance Gamma Process

Fioriani, Luciano and Semeraro [7] calibrated the Merton model including a pure-jump process, more specifically a *Variance Gamma* (VG) process, and showed that this corrected for under/overprediction of the low/high risk credit spreads. Cariboni and Schoutens [11] showed that with this method, the credit spreads become positive also for short maturities. The VG process is an example of a Lévy process of the "pure jump" type, i.e. a process that moves only by jumps, in the absence of a diffusion component. Purely discontinuous processes can be seen as time-changed Brownian motions. They admit an infinite number of (mostly small) jumps during any interval of time. In the case of a VG process, a Brownian motion with drift is time-changed with a gamma process:

Consider a gamma process $G(t; 1, \nu)$, where the parameter ν describes its shape. Let W be a standard Brownian motion. The VG process L_t^{VG} is then given by

$$L_t^{VG}(t; \sigma, \nu, \theta) = \theta G(t; 1, \nu) + \sigma W(G(t; 1, \nu)), \quad (6.28)$$

with characteristic function given by (see [11])

$$\Phi_{L_{VG}} = \mathbb{E}[\exp(iuL_t^{VG})] = (1 - iu\theta\nu + \frac{\sigma^2\nu u^2}{2})^{-\frac{1}{\nu}}. \quad (6.29)$$

Definition 6.4.1 (Variance Gamma Process). [11] *A stochastic process L^{VG} is called a VG process if it satisfies the following properties:*

1. $L_0^{VG} = 0$
2. L^{VG} has independent and stationary increments
3. For every $h > 0$, the increment $L_{t+h}^{VG} - L_t^{VG}$ follows the distribution $VG(\sigma\sqrt{t}, \frac{\nu}{t}, t\theta)$.

In order to simulate the paths of the VG process L^{VG} , the following algorithm will be utilized:

Algorithm 6.3.2 (Path of a Variance Gamma process)

Consider the time interval $[0, T]$ with n time points $0 = t_0 < \dots < t_n = T$.

1. Set $L^{VG}(0) = 0$.
2. Draw n independent standard normal random variables Z_i .
3. Draw n independent Gamma distributed random variables $\Delta\Gamma_i(\frac{\Delta t_i}{\nu}, \nu)$
4. The path of L_{VG} over the time points $0 = t_0 < \dots < t_n = T$ is then given by

$$L^{VG}(t_i) = L^{VG}(t_{i-1}) + \theta\Delta\Gamma_i + \sigma\sqrt{\Delta\Gamma_i}Z_i. \quad (6.30)$$

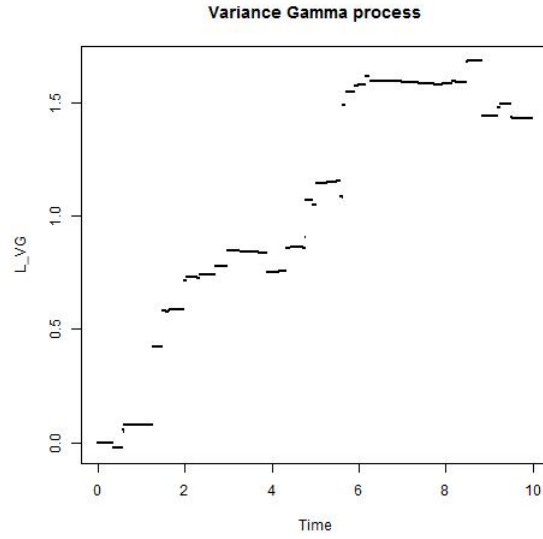


Figure 6.8: *Simulated VG process with parameters $\sigma = 0.5$, $\theta = 0.2$ and $\nu = 1.5$.*

6.4.1 Calibration to Market Data ♠

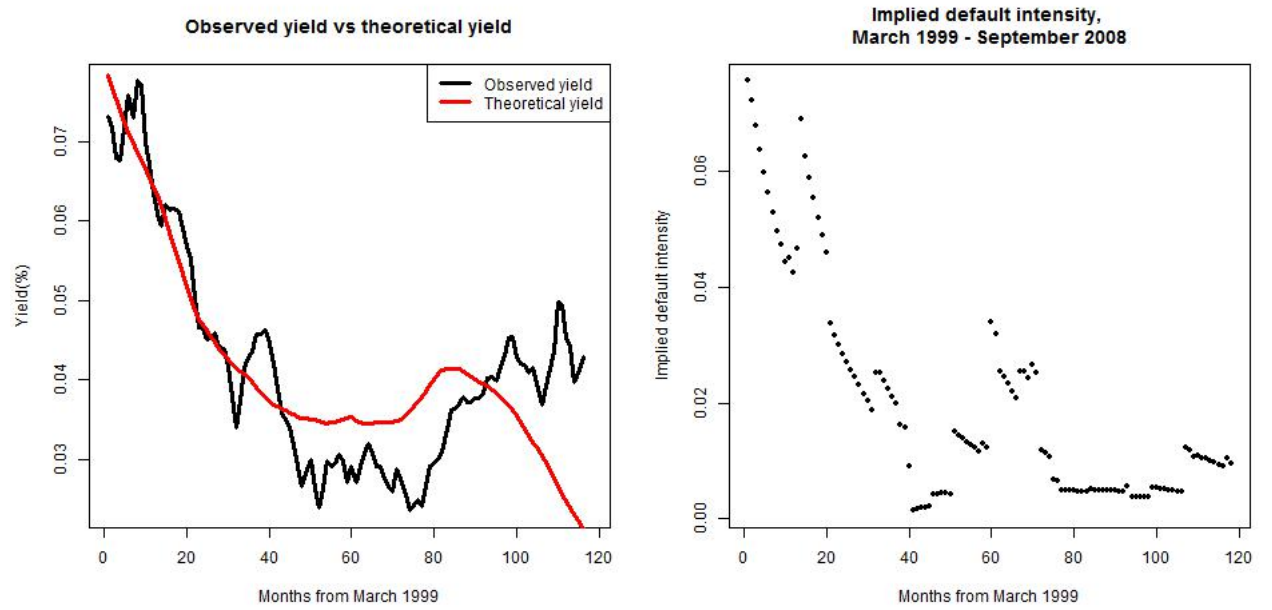


Figure 6.9: *L: Yield data vs th. yield. R: Imp. default. int. for $\theta = 10$, $\sigma = 10$, $\nu = 10$.*

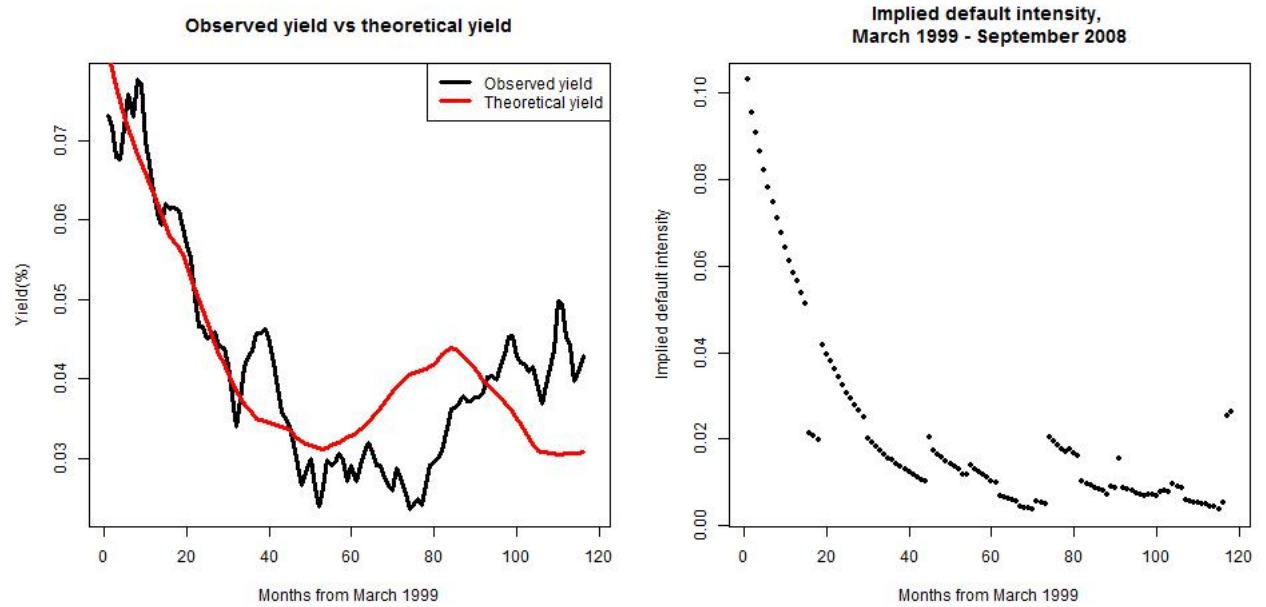


Figure 6.10: *L: Yield data vs th. yield. R: Imp. default int. for $\theta = 10, \sigma = 5, \nu = 10$.*

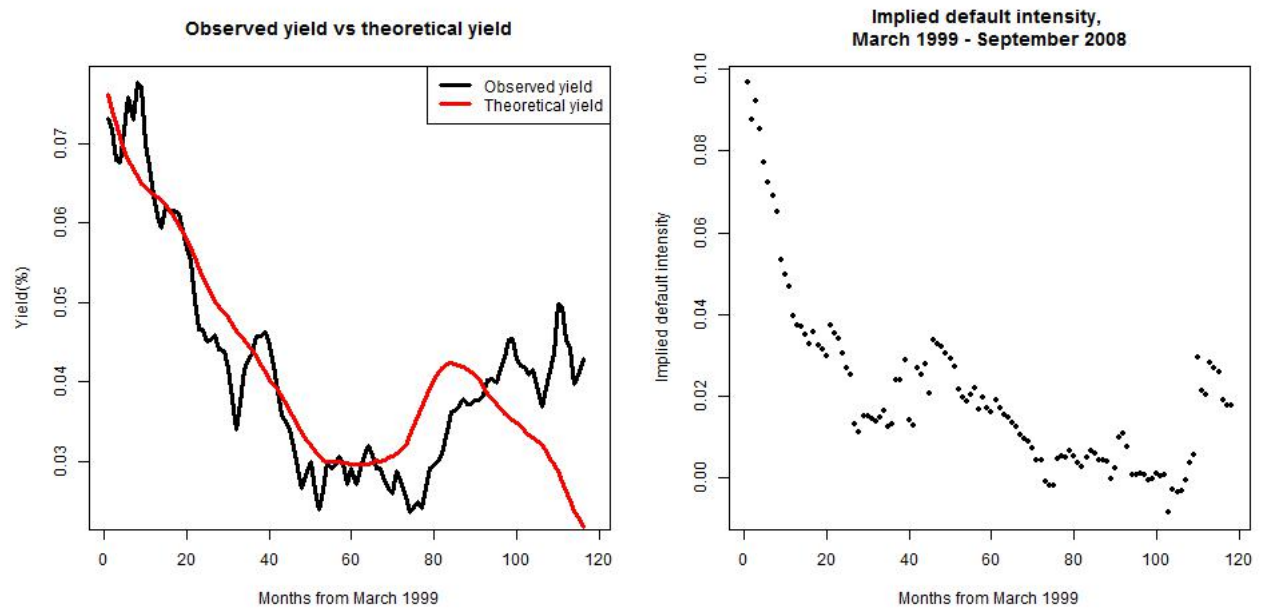


Figure 6.11: *L: Yield data vs th. yield. R: Imp. def. int. for $\theta = 0.25, \sigma = 0.75, \nu = 1.5$.*

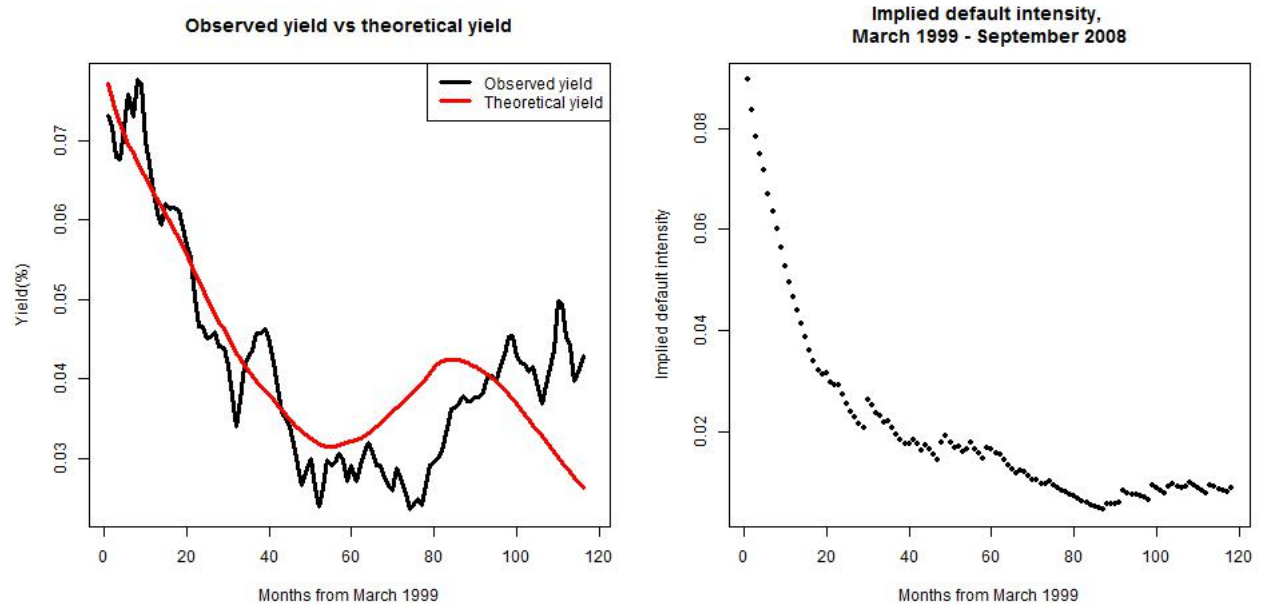


Figure 6.12: *L: Yield data vs th. yield. R: Imp. default int. for $\theta = 10$, $\sigma = 0$, $\nu = 2$.*

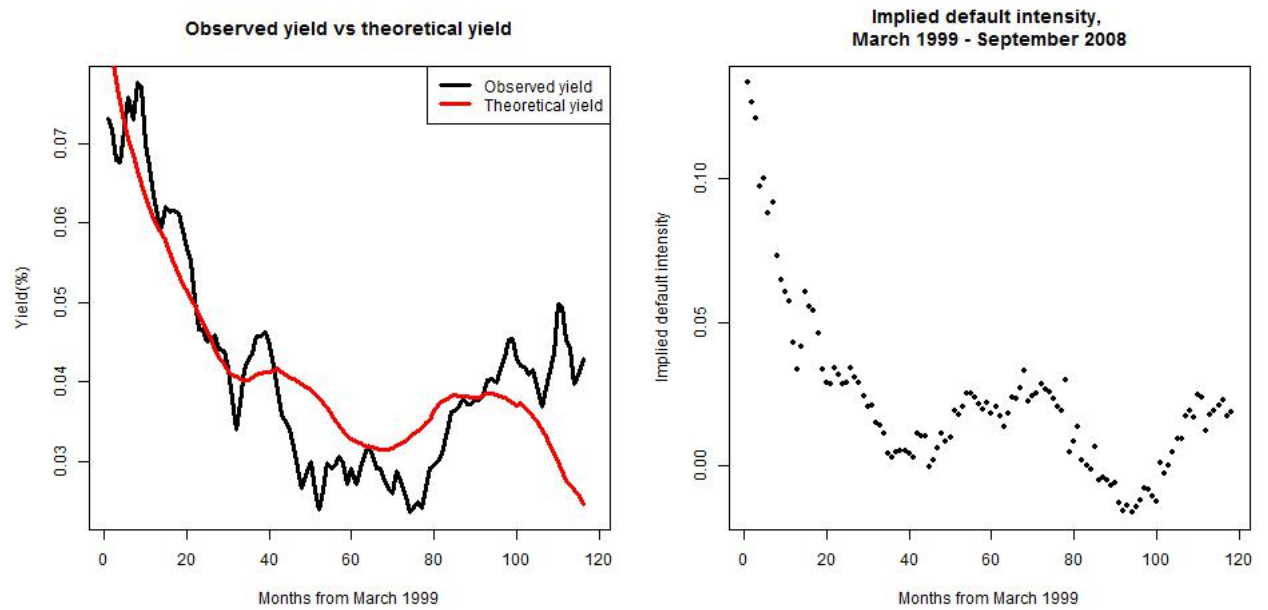


Figure 6.13: *L: Yield data vs th. yield. R: Imp. default int. for $\theta = 0$, $\sigma = 1$, $\nu = 0.5$.*

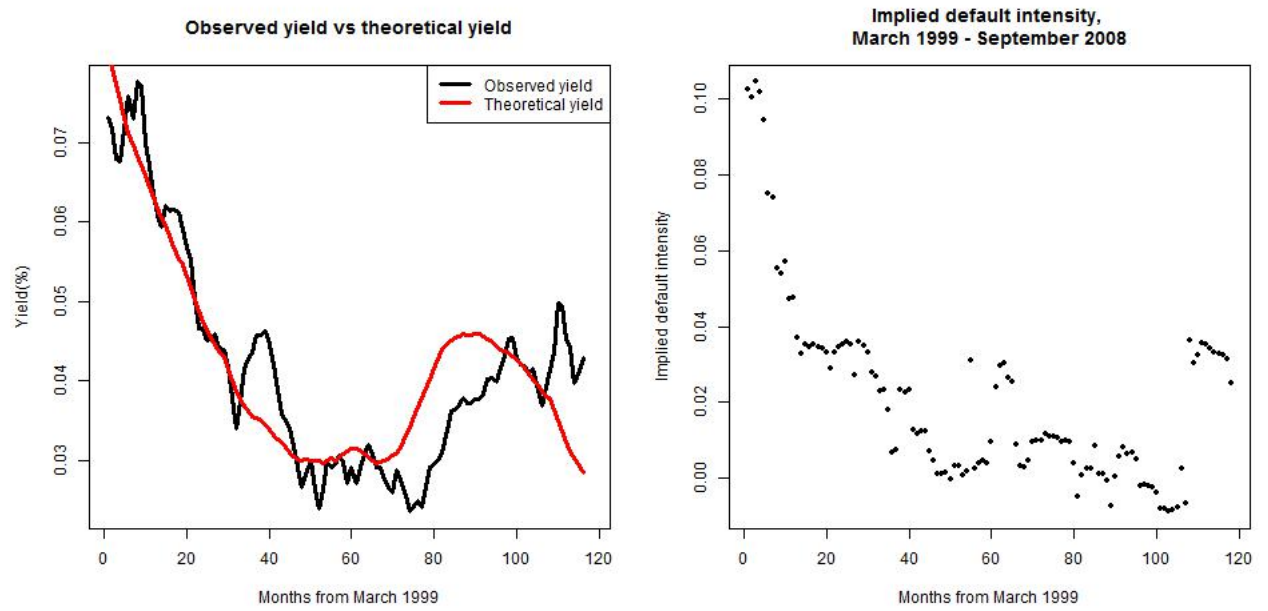


Figure 6.14: *Left: Yield data vs th. yield. Right: Imp. default int. for $\theta = 0$, $\sigma = 1$, $\nu = 1.5$.*

Chapter 7

Regime-switching Models

7.1 Motivation: Pricing of ZCBs (Calibration to Market Data) ♠

With default-free ZCBs there is no default risk involved. We are thus considering calibration of the interest rate only, in terms of the Vasicek model. The calibration will be carried out by minimizing the root mean squared error between the market yield curve data $Y_i(0, T)$ and the theoretical yield $\tilde{Y}_i(0, T)$ for all the n data points, with respect to the model parameter set $\{p\} = \{r_0, k, \theta, \sigma\}$, i.e.

$$o = \min_p \left\{ \sum_{i=1}^n \frac{(Y_i(0, T) - \tilde{Y}_i(0, T))^2}{n} \right\}^{\frac{1}{2}}. \quad (7.1)$$

The market data is taken from the Bank of Canada¹, consisting of daily yield data over the period June 1st 2006 to December 18th 2010, with a fixed three-month maturity.

We let the interest rate be given in terms of (5.8). The affine term structure involved with this type of model made it easy to calibrate, due to its closed form price expressions and thus also the yield. However, the model does not give a convincing fit to the yield curve data. The interest rate resembles a straight line due to the low resulting value of volatility and doesn't capture the relatively flat yield curve until approximately 300 trading days, which decreases rapidly until 550 trading days.

¹<http://www.bankofcanada.ca/rates/interest-rates/bond-yield-curves/>

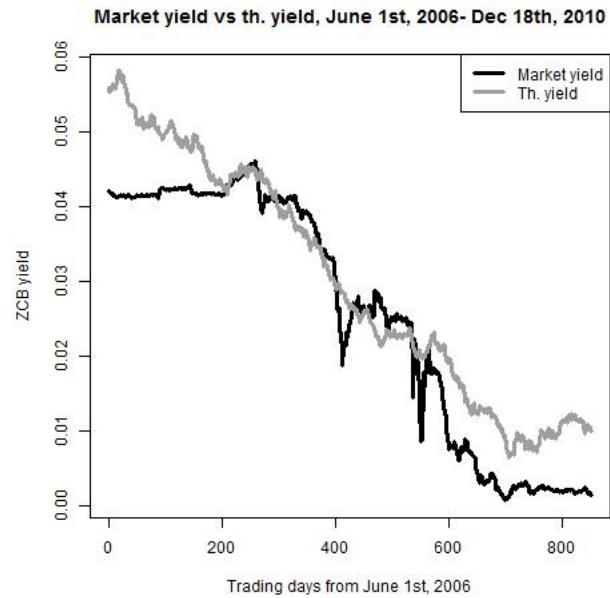


Figure 7.1: *Market yield curve (black) vs theoretical yield curve (grey).*

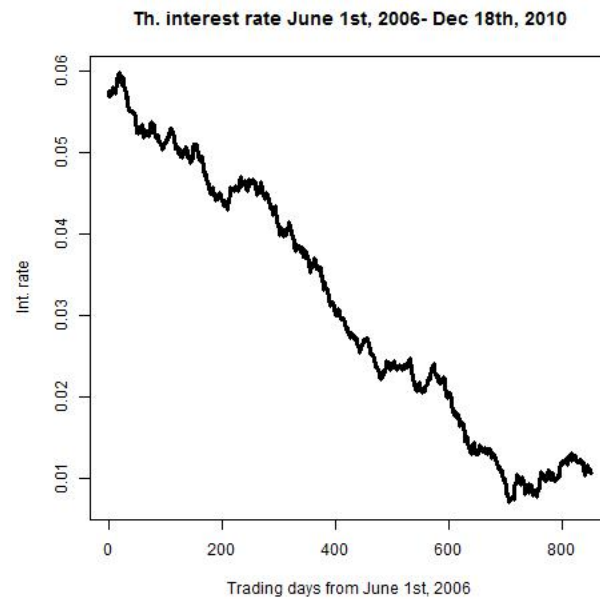


Figure 7.2: *The path of the resulting interest rate.*

7.2 Regime-switched Interest Rate and ZCB Pricing ♠

As seen in the previous section, there are cases where the Vasicek model does not provide the satisfactory results. In order to make the model more flexible, one possible modification is to replace the long term mean parameter θ by another parameter θ' once the path of r_t reaches a threshold β . This is an example of a regime-switching model: Once the interest rate reaches a certain value, one enters a new regime and the model changes. With the financial crisis of 2008 in mind, this is especially interesting.

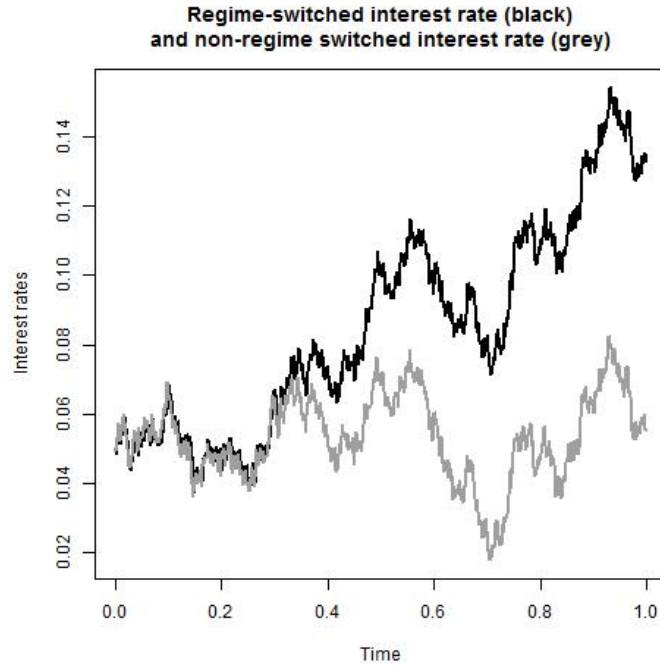


Figure 7.3: *Black: Regime-switched interest rate. Grey: Original interest rate.*
Parameters: $r_0 = 0.05, k = 1.5, \theta_1 = 0.034, \sigma = 0.065, \theta_2 = 0.15$ and $\beta = 0.06$.

In Figure 7.1, a regime-switch occurs once the interest rate path reaches $\beta = 0.06$, where it reverts towards its new long term mean $\theta_2 = 0.15$ at the rate of reversion $k = 1.5$. In comparison, the original interest rate part keeps its long term mean $\theta_1 = 0.034$.

Consider a regime-switched interest rate under the Vasicek model, i.e.

$$dr_t = k(\theta_1 \mathbb{1}_{\{r_t \geq \beta\}} + \theta_2 \mathbb{1}_{\{r_t < \beta\}} - r_t)dt + \sigma dW_t = b(r_t)dt + \sigma dW_t \quad (7.2)$$

under \mathbb{Q} . Note that the Lipschitz condition (1.21) in Theorem (1.4.1) is no longer satis-

fied due to the drift term and pricing via Euler discretization would result in large errors. However, one can apply Theorem 1.3.1 (Girsanov's theorem) in order to remove the discontinuous drift and then proceed as earlier. One obtains the representation

$$dr_t = \sigma dW_t^* \quad (\text{i.e. } r_t = r_0 + \sigma W_t^*), \quad (7.3)$$

where

$$W_t^* = W_t + \int_0^t \frac{b(r_s)}{\sigma} ds = W_t + \int_0^t \frac{b(r_0 + \sigma W_s^*)}{\sigma} ds \quad (7.4)$$

is the standard Brownian motion under \mathbb{P} .

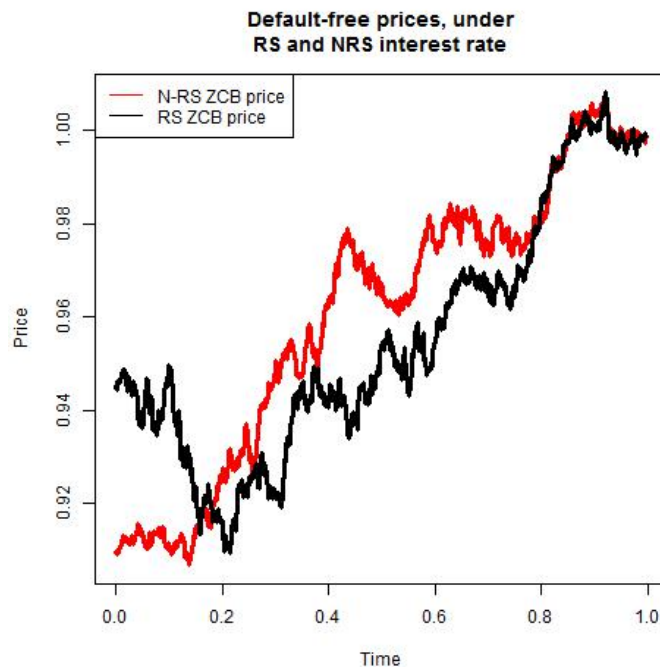


Figure 7.4: Black: Default-free ZCB prices under regime-switch.
 Grey: Default-free ZCB prices without regime-switch.
 The parameters are the same as in Fig. 7.2.

The price $B(0, T)$ of a default-free zero-coupon bond is under the regime-switch model given by

$$\mathbb{E}_{\mathbb{Q}}[\exp(-\int_0^T r_t dt)] = \mathbb{E}_{\mathbb{P}}[\exp(-\int_0^T r_t dt) M_T] =$$

$$\mathbb{E}_{\mathbb{P}}[\exp(-\int_0^T r_t dt) \exp(\frac{1}{2\sigma^2} \int_0^T b^2(r_t) dt - \frac{1}{\sigma} \int_0^T b(r_t) dW_s^*)], \quad (7.5)$$

which can be approximated by e.g. Monte Carlo methods.

7.2.1 Calibration to Market Data ♠

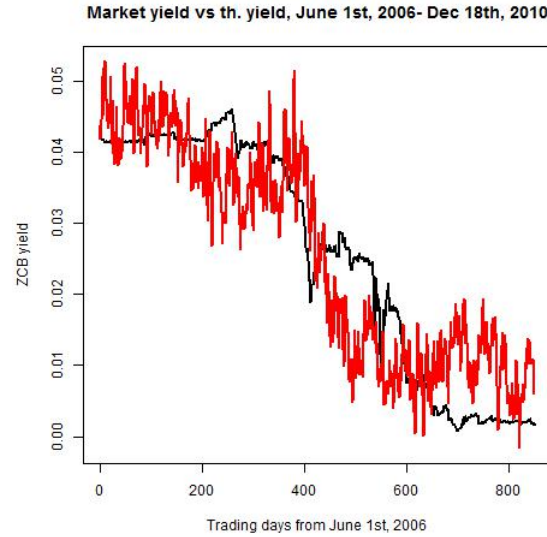


Figure 7.4: *Market yield curve (black) vs theoretical yield curve (grey).*

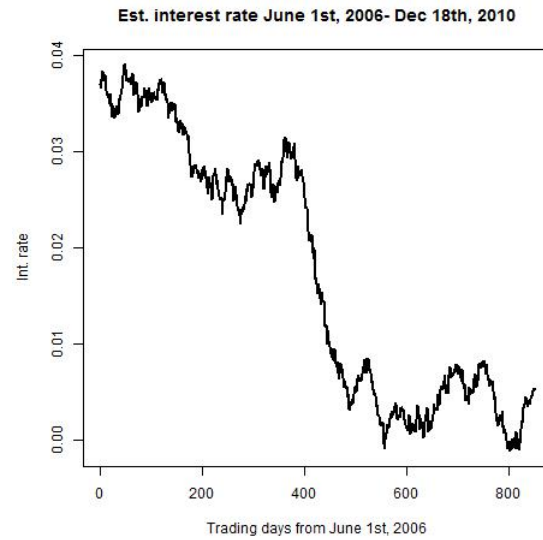


Figure 7.5: *The path of the resulting interest rate.*

7.3 Regime-switched Default Intensity and Defaultable ZCB Pricing ♠

We now want to consider a regime-switched default-risk, rather than a regime-switched interest rate and obtain expressions for the defaultable ZCB price. We will model the interest rate in terms of the original Vasicek model, i.e. let

$$dr_t = k_1(\theta_1 - r_t)dt + \sigma_1 dW_1(t) = b_1(r_t)dt + \sigma_1 dW_1(t) \quad (7.6)$$

and

$$d\gamma_t = k_2(\theta_2 \mathbb{1}_{\{\gamma_t \geq \beta\}} + \theta_3 \mathbb{1}_{\{\gamma_t < \beta\}} - \gamma_t)dt + \sigma_2 dW_2(t) = b_2(\gamma_t)dt + \sigma_2 dW_2(t), \quad (7.7)$$

describe the interest rate and the default risk, respectively. $W_1(t)$ and $W_2(t)$ are two independent Brownian motions. In matrix form,

$$\begin{pmatrix} dr_t \\ d\gamma_t \end{pmatrix} = \begin{pmatrix} b_1(r_t) \\ b_2(\gamma_t) \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}.$$

Following Girsanov's theorem again, the representation for the interest rate and default risk becomes

$$\begin{pmatrix} dr_t \\ d\gamma_t \end{pmatrix} = \begin{pmatrix} \sigma_1 dW_1^*(t) & 0 \\ 0 & \sigma_2 dW_2^*(t) \end{pmatrix},$$

where

$$\begin{pmatrix} dW_1^*(t) \\ dW_2^*(t) \end{pmatrix} = \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix} + \begin{pmatrix} \frac{b_1(r_t)}{\sigma_1} \\ \frac{b_2(\gamma_t)}{\sigma_2} \end{pmatrix} dt.$$

Following the same arguments as for the default-free ZCB, the price of a defaultable bond is found to be

$$\begin{aligned} B^d(0, T) &= E_{\mathbb{Q}}[\exp(-\int_0^T (r_t + \gamma_t)dt)] = B(0, T)E_{\mathbb{Q}}[\exp(-\int_0^T \gamma_t dt)] = \\ &= B(0, T)E_{\mathbb{P}}[\exp(-\int_0^T \gamma_t dt) \exp(\frac{1}{2} \int_0^T \frac{b_2^2(\gamma_t)}{\sigma_2^2} dt - \int_0^T \frac{b_2(\gamma_t)}{\sigma_2} dW_2^*(t))], \end{aligned} \quad (7.8)$$

where

$$B(0, T) = E_{\mathbb{P}}[\exp(-\int_0^T r_t dt) \exp(\frac{1}{2} \int_0^T \frac{b_1^2(r_t)}{\sigma_1^2} dt - \int_0^T \frac{b_1(r_t)}{\sigma_1} dW_1^*(t))] \quad (7.9)$$

describes the default-free bond price.

7.4 Regime-switched Correlated Default Intensity ♠

It is natural to incorporate correlation between the interest rate and the default risk. Let $\rho \in [-1, 1]$ denote the correlation between r_t and γ_t . Furthermore, let

$$dr_t = k_1(\theta_1 - r_t)dt + \sigma_1 dW_1(t) = b_1(r_t)dt + \sigma_1 dW_1(t) \quad (7.10)$$

and

$$\begin{aligned} d\gamma_t &= k_2(\theta_2 \mathbb{1}_{\{\gamma_t \geq \beta\}} + \theta_3 \mathbb{1}_{\{\gamma_t < \beta\}} - \gamma_t)dt + \sigma_2(\rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t)) = \\ &= b_2(\gamma_t)dt + \sigma_2(\rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t)), \end{aligned} \quad (7.11)$$

describe the interest rate and the default risk, respectively. $W_1(t)$ and $W_2(t)$ are two independent Brownian motions. In matrix form,

$$\begin{pmatrix} dr_t \\ d\gamma_t \end{pmatrix} = \begin{pmatrix} b_1(r_t) \\ b_2(\gamma_t) \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2 \rho & \sigma_2 \sqrt{1 - \rho^2} \end{pmatrix} \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix} = \tilde{b}(r_t, \gamma_t)dt + \Sigma dW(t). \quad (7.12)$$

We find the new representation of the default risk and the interest rate by first solving for θ :

$$\begin{aligned} \theta(t, \omega) &= \Sigma^{-1} \tilde{b}(r_t, \gamma_t) = \\ &= \frac{1}{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \begin{pmatrix} \sigma_2 \sqrt{1 - \rho^2} & 0 \\ -\sigma_2 \rho & \sigma_1 \end{pmatrix} \begin{pmatrix} b_1(r_t) \\ b_2(\gamma_t) \end{pmatrix} = \\ &= \begin{pmatrix} \frac{1}{\sigma_1} & 0 \\ -\frac{\rho}{\sigma_1 \sqrt{1 - \rho^2}} & \frac{1}{\sigma_2 \sqrt{1 - \rho^2}} \end{pmatrix} \begin{pmatrix} b_1(r_t) \\ b_2(\gamma_t) \end{pmatrix}, \end{aligned}$$

which gives

$$\begin{pmatrix} dr_t \\ d\gamma_t \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2 \rho & \sigma_2 \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} dW_1^*(t) \\ dW_2^*(t) \end{pmatrix}, \quad (7.13)$$

where the \mathbb{P} -Brownian motions are given as

$$\begin{pmatrix} dW_1^*(t) \\ dW_2^*(t) \end{pmatrix} = \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{\sigma_1} & 0 \\ -\frac{\rho}{\sigma_1 \sqrt{1-\rho^2}} & \frac{1}{\sigma_2 \sqrt{1-\rho^2}} \end{pmatrix} \begin{pmatrix} b_1(r_t) \\ b_2(\gamma_t) \end{pmatrix} dt. \quad (7.14)$$

The price of the defaultable zero-coupon bond can then be found by the same types of calculations as in (7.5) and (7.8).

Conclusion

From the calibration results, it is evident that the Gaussian Vasicek model gives the worst fit for the yield curves, which is consistent with the initial idea that one should incorporate heavy tailed distributions in order to account for extreme events. Both the α -stable and Variance Gamma process result in better fits than Brownian motion, when used as driving processes for the Vasicek model. An idea for further investigation could be to examine the case of an α -stable process in the case of other credit derivatives, such as credit default swaps. W. Schoutens and J. Cariboni [11] calibrated a general Ornstein-Uhlenbeck process driven by a Variance Gamma process to the CDS term structures of various companies with great success. The drawback of utilizing Monte Carlo methods in these types of models (or in general) is the slow computational speed. P. Carr and D. Madan [4] utilized instead the Fast Fourier Transform (FFT) in order to price options with dynamics described by jump processes. One could attempt to follow a similar routine in order to price credit derivatives more efficiently.

As intuitively expected (due to the extra 'freedom' of switching the mean reversion constant), the regime-switching model improves the fit for the yield curves of the default-free ZCB prices, when calibrated to market data. Unfortunately, there was not enough time to fit the regime-switch model in the case of defaultable bonds, as it is a potentially interesting problem. As in the case of α -stable and the Variance Gamma process, another idea could be to price other credit derivatives under this new model. Although potentially complicated, a further extension could be combining regime-switching and jump processes.

Chapter 8

Appendix

8.1 Solution of the Gaussian Vasicek model

The Vasicek model is given by

$$d\gamma_t = k(\theta - \gamma_t)dt + \sigma dW_t.$$

By Ito's lemma with $f(t, x) = e^{kt}x$ where $f(t, \gamma_t) = e^{kt}\gamma_t$, we obtain

$$\begin{aligned} d(f(t, \gamma_t)) &= ke^{kt}\gamma_t dt + e^{kt}d\gamma_t = ke^{kt}\gamma_t dt + e^{kt}(k(\theta - \gamma_t)dt + \sigma dW_t) = \\ &= k\theta e^{kt}dt + \sigma e^{kt}dW_t. \end{aligned}$$

This is equivalent to

$$e^{kt}\gamma_t = \gamma_0 + k\theta \int_0^t e^{ks}ds + \sigma \int_0^t e^{ks}dW_s,$$

solving for γ_t then yields

$$\begin{aligned} \gamma_t &= \gamma_0 e^{-kt} + k\theta e^{-kt} \int_0^t e^{ks}ds + \sigma e^{-kt} \int_0^t e^{ks}dW_s = \\ &= \gamma_0 e^{-kt} + k\theta e^{-kt} \left[\frac{1}{k} e^{ks} \right]_0^t + \sigma e^{-kt} \int_0^t e^{ks}dW_s. \end{aligned}$$

Finally,

$$\gamma_t = \gamma_0 e^{-kt} + \theta(1 - e^{-kt}) + \sigma e^{-kt} \int_0^t e^{ks} dW_s.$$

8.2 Solution of the Lévy-driven Vasicek model

$$d\gamma_t = k(\theta - \gamma_t)dt + \sigma dL_t.$$

By Proposition 6.2.1, with $f(t, x) = e^{kt}x$,

$$\begin{aligned} f(t, \gamma_t) - f(0, \gamma_0) &= \int_0^t k e^{ks} \gamma_s ds + \int_0^t e^{ks} d\gamma_s + \frac{1}{2} \cdot 0 + \\ &\quad \sum_{0 \leq s \leq t, \Delta \gamma_t \neq 0} [e^{ks} \gamma_s - e^{ks} \gamma_{s-} - \Delta \gamma_s e^{ks}] = \\ &\quad \int_0^t (k e^{ks} \gamma_s + e^{ks} (k(\theta - \gamma_s))) ds + \int_0^t \sigma e^{ks} dL_s + \\ &\quad \sum_{0 \leq s \leq t, \Delta \gamma_t \neq 0} [e^{ks} \gamma_s - e^{ks} \gamma_{s-} - e^{ks} (\gamma_s - \gamma_{s-})] = \\ &\quad \int_0^t k \theta e^{ks} ds + \sigma \int_0^t e^{ks} dL_s = \\ &\quad \theta(e^{kt} - 1) + \sigma \int_0^t e^{ks} dL_s, \end{aligned}$$

hence

$$\gamma_t = \gamma_0 e^{-kt} + \theta(1 - e^{-kt}) + \sigma \int_0^t e^{k(s-t)} dL_s.$$

Chapter 9

Tables

Calibrated parameters of the Greek defaultable bonds (α -stable proceses)

| Objective value | γ_0 | k | θ | σ | α used |
|-----------------|------------|--------|----------|----------|---------------|
| 0.000071 | 0.0850 | 6.6534 | 0.0105 | -0.0990 | 2 |
| 0.000070 | 0.0860 | 10 | 0.0072 | -0.1660 | 1.5 |
| 0.000065 | 0.1018 | 10 | 0.0094 | 0.0618 | 1 |
| 0.000028 | 0.1125 | 10 | 0.0015 | 0.0041 | 0.5 |

Calibrated parameters of the Greek defaultable bonds (Variance Gamma process)

| Objective value | θ_{VG} used | σ_{VG} used | ν used | γ_0 | k | θ | σ |
|-----------------|--------------------|--------------------|------------|------------|----------|----------|----------|
| 0.000067 | 10 | 10 | 10 | 0.075597 | 9.993227 | 0.004515 | 0.392124 |
| 0.000063 | 10 | 5 | 10 | 0.102892 | 7.385951 | 0.000001 | 1.242857 |
| 0.000057 | 0,25 | 0,75 | 1,5 | 0.096635 | 9.999440 | 0.002176 | 1.018828 |
| 0.000054 | 10 | 0 | 2 | 0.089618 | 9.999900 | 0.000012 | 0.046113 |
| 0.000053 | 0 | 1 | 0,5 | 0.133464 | 9.998035 | 0.013717 | 0.353567 |
| 0.000045 | 0 | 1 | 1,5 | 0.102339 | 3.536868 | 0.000002 | 0.993964 |

Calibrated parameters of the Canadian ZCBs (Vasicek)

| Objective value | r_0 | k | θ | σ |
|-----------------|-------|--------|----------|----------|
| 0.0051 | 0.058 | 0.1635 | -0.2290 | 0.0116 |

Calibrated parameters of the Canadian ZCBs (Regime-switch)

| Objective value | r_0 | k | θ_1 | σ | θ_2 | β |
|-----------------|------------|------------|-------------|------------|------------|-------------|
| 0.000063 | 0.03701598 | 0.10784701 | -0.17126976 | 0.16723585 | 0.06619342 | 0.027149957 |

Chapter 10

R Code

Simulation of Brownian motion (Fig. 1.1)

```
n <- 1000 #Number of time points
dt <- 1/n #Time step
t <- seq(0,1,dt) #Time points

#Gaussian increments:
dW <- rnorm(n,0,1)*sqrt(dt)

#Brownian motion:
W <- rep(0,n)

for(i in 2:n){
    W[i] <- W[i-1] + dW[i-1]
}

#Plot:
jpeg('BM.jpg')
plot(t[1:n],W, type="l", lwd=2, main = "Brownian motion W as
a function of time", xlab="Time", ylab = "W_t")
dev.off()
```

Simulation of default intensity following the Vasicek model (Fig. 3.1)

```
#Parameter values:
k <- 0.3
theta <- 0.1
sigma <- 0.03

#Initial value:
gamma0 <- 0.05

n <- 1000
dt <- 1/n
t = seq(0,1,dt)

gamma <- rep(0,n)
gamma[1] <- gamma0 #Initial value
for(i in 2:n){ #Euler discretization of gamma:
    gamma[i] <- gamma[i-1]+k*(theta-gamma[i-1])*dt+
                sigma*rnorm(1,0,1)*sqrt(dt)
}

jpeg('DefaultIntensity.jpg')
plot(t[1:n], gamma, type = 'l', lwd=2, xlab = "Time, t", ylab = "gamma_t",
main = "Default Intensity (gamma_t)")
dev.off()
```

Simulation of bond prices with correlated interest rate and default intensity (Fig. 3.2)

```
T <- 1 #Maturity
n <- 1000 #No. of time points
dt <- 1/n #Time step
t <- seq(0,T, dt) #Time points
```

```

#Model parameters:

#Interest rate:
r0 <- 0.03
k <- 0.2
theta <- 0.1
sigma <- 0.02

#Default intensity:
gamma0 <- 0.05
k <- 0.2
u <- 0.2
tau <- 0.05
rho <- 0.2

#Vector for interest rate r
r <- rep(0,n)
#Vector for default intensity gamma:
gamma <- rep(0,n)

#Initial values:
r[1] <- r0
gamma[1] <- gamma0

#Brownian increments dW_t^1 and dW_t^2:
dW1 <- rnorm(n,0,1)*sqrt(dt)
dW2 <- rnorm(n,0,1)*sqrt(dt)

#Euler discretization of (3.63):
for(i in 2:n){
    r[i] <- r[i-1]+k*(theta-r[i-1])*dt+sigma*dW1[i]
}

#Euler discretization of (3.73):
for(i in 2:n){
    gamma[i] <- k*(u-gamma[i-1])*dt+tau*
    (rho*dW1[i]+sqrt(1-rho**2)*dW2[i])
}

```

```

#Affine function a(t,T) (3.84):
a <- rep(0,n)
for(i in 1:n){
  a[i] <- ((gamma[i]+r[i])/k)*(1-exp(-k*(T-t[i])))
          +((u+theta)/(k))*(k*(T-t[i])+exp(-k*(T-t[i]))-1)
}

#Affine function b^2(t,T) (3.85):
b2 <- rep(0,n)
for(i in 1:n){
  b2[i] <- ((sigma**2+tau**2)/(k**2))*(T-t[i]+(1/(2*k))*
          (4*exp(-k*(T-t[i]))-exp(-2*k*(T-t[i]))-3)+((sigma*tau*rho)/k)*
          (1-exp(-2*k*(T-t[i]))))
}

#Vector for default-free bond price:
B <- rep(0,n)
#Vector for defaultable bond price:
Bd <- rep(0,n)

for(i in 1:n){
  Bd[i] <- exp(0.5*b2[i]-a[i])
}

#Default-free case (int.rate only):

#Affine function A(t,T):
A <- rep(0,n)
for(i in 1:n){
  A[i] <- (r[i]/k)*(1-exp(-k*(T-t[i]))) +
          (theta/k)*(k*(T-t[i])+exp(-k*(T-t[i]))-1)
}

```

```

#Affine function C(t,T):
C <- rep(0,n)
for(i in 1:n){
  C[i] <- ((sigma**2)/(k**2))*(T-t[i]+(1/(2*k))*
    (4*exp(-k*(T-t[i]))-exp(-2*k*(T-t[i]))-3))
}

for(i in 1:n){
  B[i] <- exp(0.5*C[i]-A[i])
}

#Plot:
y1 <- min(Bd[2:n],B[2:n])
y2 <- max(Bd[2:n],B[2:n])

jpeg('DefaultableBondPrice.jpg')
plot(t[2:n+1], Bd[2:n+1], type="l", lwd=1.5, xlab="Time", ylab="Price",
main="Defaultable ZCB price", ylim=c(y1,y2))
dev.off()
jpeg('DefaultfreeBondPrice.jpg')
plot(t[2:n+1], B[2:n+1], type="l", lwd=1.5, xlab="Time", ylab="Price",
main="Default-free ZCB price",ylim=c(y1,y2))
dev.off()

```

Simulation of symmetric alpha-stable processes (Fig. 6.1)

```

#Based on Algorithm 6.3.1.
#(Repeated for alpha = 0.1, 0.5, 1 and 1.9)

#1: Simulate n ind rv. gamma_i, uniformly distributed on (-pi/2,pi/2):
# and n independent exponential r.v.'s W_i:

n = 5000
gamma <- runif(n, -pi/2, pi/2)
W <- rexp(n, rate = 1)

#2: Compute delta X_i for i = 1,...,n using 6.4 with t_0 = 0:

alpha <- 1.9
dt <- 1/n
t <- seq(0, 1, dt)

```



```

delta <- rep(0,n)
for(i in 2:n){
    delta[i] <- ((t[i]-t[i-1])^(1/alpha) )*(sin(gamma[i]*alpha)/
    ((cos(gamma[i]))^(1/alpha)))*((cos((1-alpha)*gamma[i])/W[i])
    ^((1-alpha)/alpha))
}

#3 The discretized trajectory is given by  $X(t_i) = \sum_{k=1}^i \text{delta } X_k$ 

X <- rep(0,n)
for(i in 2:n){
    X[i] <- X[i-1]+delta[i]
}

#Plot
jpeg('ASTP19.jpg')
    plot(t[1:n],X, cex = 0.3, xlab = "Time", ylab = "X_t",
    main = "Alpha-stable process with alpha = 1.9")
dev.off()

```

Simulation of Levy-driven Vasicek model (Fig. 6.2)

```

#Parameter values:
k <- 0.2
theta <- 0.15
sigma <- 0.05
alpha <- 1.5

#Initial value:
gamma0 <- 0.05

n <- 10000
dt <- T/n
t <- seq(0,n,dt)

gamma <- rep(0,n) #Def. intensity.
Levy <- rep(0,n)
gamma[1] <- gamma0

```

```

#Returns increments:
L <- function(alpha){
  gd <- runif(1, -pi/2, pi/2)
  W1 <- rexp(1, rate = 1)
  delta <- ((dt)^(1/alpha) )*(sin(gd*alpha)/
((cos(gd))^(1/alpha)))*((cos((1-alpha)*gd)/W1)^
((1-alpha)/alpha))
  return(delta)
}

#The increments for the given alpha:
for(i in 1:n){
  Levy[i] <- L(alpha)
}

#The resulting default intensity:
for(i in 2:n){
  gamma[i] <- gamma[i-1]+k*(theta-gamma[i-1])*dt+sigma*Levy[i]
}

#Plot:
jpeg('VasicekJump.jpg')
plot(t[1:n], gamma, cex=.1, main="Vasicek model driven by alpha-stable process",
xlab="Time", ylab="gamma_t")
dev.off()

```

Code for constructing the survival curve given by Levy driven Vasicek model (Fig. 6.3)

```

#No. of Monte Carlo computations:
N <- 10000
#No. of time points:
n <- 1000
#Time step:
dt <- 1/n
#Time points:
t <- seq(dt, 1, dt)

```

```

#Construction of Levy process (n x N matrix)
#(following Algorithm 6.3.1):

alpha = 1.5

set.seed(123)
gd <- matrix(runif(n*N, -pi/2, pi/2), nrow = n, ncol=N)
set.seed(123)
W1 <- matrix(rexp(n*N, rate = 1), nrow = n, ncol = N)

Levy <- mat.or.vec(n,N)

for(j in 1:N){
  for(i in 1:n){
    Levy[i,j] <- ((dt)^(1/alpha) )*(sin(gd[i,j]*alpha)/
      ((cos(gd[i,j]))^(1/alpha)))*((cos((1-alpha)*gd[i,j])/W1[i,j])^
      ((1-alpha)/alpha))
  }}

#Parameter values:
gamma0 = 0.05
k <- 0.2
theta <- 0.15
sigma <- 0.05

deftimes <- rep(0,N)
defcount <- rep(0,N)

#Bootstrapping of the survival curve:

U <- runif(N, min=0, max=1)
for(j in 1:N){
  gamma <- rep(0,n)
  gamma[1] <- gamma0
  lambda <- rep(0,n)
  incr <- rep(0,n)
  lambda[1] <- 1

```

```

    for(i in 2:n){
      gamma[i] <- gamma[i-1]+k*(theta-gamma[i-1])*dt+sigma*Levy[i,j]
      incr[i] <- gamma[i]-gamma[i-1]
      lambda[i] <- lambda[i-1]*exp(-incr[i])
      if(U[j]>lambda[i]){
        deftimes[j] <- i*dt
        defcount[j] <- 1
        break
      }
      else{deftimes[j] <- -1}
    }
  }

P <- function(x){
  1-(length(deftimes[deftimes<x])/N)
}

#Survival probabilities:
Surv <- rep(0,n)
for(i in 1:n){
  Surv[i] <- P(t[i])
}

#Plot:
jpeg('SurvProp.jpg')
plot(t, Surv, main="Survival probabilities", xlab="Time, t")
dev.off()

```

Calibration of defaultable ZCBs with symmetric α -stable processes (Fig. 6.4-6.7)

```

#Load the ECB interest rates.
int.rate <- read.table('euro.txt')
int.rate <- as.matrix(int.rate)
int.rate <- as.numeric(int.rate)
int.rate <- int.rate*(1/100)
int.rate <- int.rate[3:156]

#Yield data with 3 year maturity:
Yield1 <- read.table('Greek.txt', header = FALSE)
Yield1 <- Yield1[,4]
Yield1 <- Yield1/100

```

```

Yield1 <- as.matrix(Yield1)
Yield <- as.numeric(Yield1)
Yield <- rev(Yield)

#The following is repeated for
#alpha=2, 1.5, 1, 0.5:
#Repeated for the different values of alpha:
#(following Algorithm 6.3.1):

N <- 1000 #No. of MC repetitions
gd <- matrix(runif(n*N, -pi/2, pi/2), nrow = n, ncol=N)
W1 <- matrix(rexp(n*N, rate = 1), nrow = n, ncol = N)

Levy <- mat.or.vec(n,N)

for(j in 1:N){
  for(i in 1:n){
    Levy [i,j] <- ((dt)^(1/alpha) )*(sin(gd[i,j]*alpha)/
      ((cos(gd[i,j]))^(1/alpha)))*((cos((1-alpha)*gd[i,j])/W1[i,j])^
      ((1-alpha)/alpha))
  }}

#Function to minimize:
f <- function(k){
  g[1,] <- k[1]
  for(j in 1:N){
    for(i in 2:n){
      g[i,j] <- g[i-1,j]+k[2]*{k[3]-g[i-1,j]}*dt+k[4]*Levy[i,j]
    }
    for(j in 1:N){
      for(i in 1:(n-tau)){ #Th. yield:
        Y[i,j] <- {sum(r[i:(i+tau)])+sum(g[i:(i+tau),j])}/{tau+1}
      }
      sum((Yield-rowMeans(Y))*2/(n-tau)) #Obj. function
    }
  }

  o <- DEoptim(f,c(0,0,0,-1),c(1,10,2,4),control = DEoptim.control(trace = 1,
    strategy = 1, itermax = 10000,steptol = 100, reltol = 1e-3))
  p <- o$optim$bestmem

  g <- mat.or.vec(n,N)

```

```

Y <- mat.or.vec(n-tau,N)

k <- p

g[1,] <- k[1]

for(j in 1:N){
  for(i in 2:n){
    g[i,j] <- g[i-1,j]+k[2]*(k[3]-g[i-1,j])*dt+k[4]*Levy[i,j]
  }}

for(j in 1:N){
  for(i in 1:(n-tau)){
    Y[i,j] <- (sum(r[i:(i+tau)]))+sum(g[i:(i+tau),j])/(tau+1)
  }}

Yr <- rowMeans(Y)
Yr1 <- Yr
gr <- rowMeans(g)
gr1 <- gr

plot(Yield[1:118], type="l", lwd=3, main="Observed yield vs theoretical yield",
xlab="Months from March 1999", ylab="Yield(%)")
lines(Yr1, col="red", lwd=3)
legend(x="topright", legend =c("Observed yield", "Theoretical yield"),
col=c("black", "red"), lwd=3)

#Plot the implied default probabilities:
plot(gr1[1:118], cex=.1,lwd=3, main="Implied default intensity,
March 1999 - September 2008", xlab="Months from March 1999",
ylab="Implied default intensity")

Simulation of Variance Gamma process (Fig. 6.8)

#End point:
T <- 1
#No. of time points:
n = 1000

```

```

#Time step:
dt = 1/n
#Time points:
t <- seq(0,T,dt)

#Vector for VG process:
X <- rep(0,n)

#Model parameters:
theta = 0.2
sigma = 0.5
nu = 1.5

#Following algorithm 6.3.2:
DG <- rgamma(n, dt/nu, rate = nu)
Z <- rnorm(n,0,1)

#Discretized path of VG process:
for(i in 2:n){
    X[i] <- X[i-1]+theta*DG[i]+sigma*sqrt(DG[i])*Z[i]
}

#Plot path:
plot(t[1:n], X, type="l", main="Variance Gamma process", xlab="Time", ylab="L_VG")

Calibration of defaultable ZCBs with Variance Gamma process (Fig. 6.9-6.14)

#Use the same yield and interest rate data as for the alpha stable case
N <- 1000 #No. of MC repetitions
tau <- 36 #No of dates to integrate
Yield <- Yield[1:(n-tau)]

g <- mat.or.vec(n,N) #Default intensity
Y <- mat.or.vec(n-tau,N) #Yield

#The following is repeated for different
#values of theta, sigma and nu:
#(following Algorithm 6.3.2):

DG <- matrix(rgamma(n*N, dt/nu, rate = nu), nrow = n)
Z <- matrix(rnorm(n*N,0,1), nrow=n)

```

```

#Variance Gamma process given parameters theta, sigma, nu:
VG <- theta*DG+sigma*sqrt(DG)*Z

#Function to minimize:
f <- function(k){
  g[1,] <- k[1]
  for(j in 1:N){
    for(i in 2:n){ #Default intensity:
      g[i,j] <- g[i-1,j]+k[2]*{k[3]-g[i-1,j]}*dt+k[4]*VG[i,j]
    }
    for(j in 1:N){
      for(i in 1:(n-tau)){ #Th. yield:
        Y[i,j] <- {sum(r[i:(i+tau)])+sum(g[i:(i+tau),j])}/{tau+1}
      }
    }
    #Objective function:
    sum((Yield-rowMeans(Y))*2/(n-tau))
  }
}

o <- DEoptim(f,c(0,0,0,0),c(1,10,2,4),control = DEoptim.control(trace = 1,
strategy = 1, itermax = 10000,steptol = 100, reltol = 1e-3))
#Save calibrated parameters:
p <- as.numeric(o$optim$bestmem)

#Calculate default intensity and
#yield given the calibrated parameters:

g <- mat.or.vec(n,N)
Y <- mat.or.vec(n-tau,N)
k <- p
g[1,] <- k[1]

for(j in 1:N){
  for(i in 2:n){
    g[i,j] <- g[i-1,j]+k[2]*(k[3]-g[i-1,j])*dt+k[4]*VG[i,j]
  }
}

```



```

for(j in 1:N){
for(i in 1:(n-tau)){
      Y[i,j] <- (sum(r[i:(i+tau)])+sum(g[i:(i+tau),j]) )/(tau+1)
}}

Yr <- rowMeans(Y)
Yr1 <- Yr #Save calibrated yield
gr <- rowMeans(g)
gr1 <- gr #Save calibrated default intensity

#Produce plots:

plot(Yield[1:118], type="l", lwd=3, main="Observed yield vs theoretical yield",
xlab="Months from March 1999", ylab="Yield(%)"
lines(Yr1, col="red", lwd=3)
legend(x="topright", legend =c("Observed yield", "Theoretical yield"),
col=c("black", "red"), lwd=3)

#Plot the implied default probabilities:
plot(gr1[1:118], cex=.1,lwd=3, main="Implied default intensity,
March 1999 - September 2008", xlab="Months from March 1999",
ylab="Implied default intensity")

```

Code for calibrating default-free ZCBs (via Vasicek): (Fig. 7.1-7.2)

```

#Import yield data:
Yield <- dget('ThYield0.25.txt')

n <- length(Yield) #No of dates to consider
dt <- 1/n          #Time step

W <- rnorm(n,0,1) #Brownian motion
r <- rep(0,n) #Interest rate
Y <- rep(0,n) #Yield

#Set up objective function:
f <- function(k){
      r[1] <- k[1]
      for(i in 2:n){
            #Vasicek model for interest rate:
            r[i] <- r[i-1]+k[2]*(k[3]-r[i-1])*dt+k[4]*W[i]*sqrt(dt)

```

```

    }
    #Affine constant nr 1:
    A <- (1/k[2])*(1-exp(-k[2]*T))
    #Affine constant nr 2:
    B <- (k[3]-k[4]**2/(2*k[2]))*(A-T)-
        (k[4]**2/(4*k[2]))*A**2
    #The theoretical yield:
    for(i in 1:n){
    Y[i] <- (A*r[i]-B)/T
    }
    sqrt(sum((Yield-Y)**2/n)) #Obj. function to minimize
}

#Minimize wrt parameters:
h <- DEoptim(c(0,-1,-1,-1),c(1,1,1,1),f)

#Save the calibrated parameters:
p <- as.numeric(h$optim$bestmem)

#Theoretical model given calibrated parameters:
r <- rep(0,n)
Y <- rep(0,n)

#Euler discretization of interest rate:
r[1] <- p[1]
for(i in 2:n){
    r[i] <- r[i-1]+p[2]*(p[3]-r[i-1])*dt+p[4]*W[i]*sqrt(dt)
}

#Affine constant nr 1:
A <- (1/p[2])*(1-exp(-p[2]*T))
#Affine constant nr 2:
B <- (p[3]-p[4]**2/(2*p[2]))*(A-T)-
    (p[4]**2/(4*p[2]))*A**2

#Theoretical yield:
for(i in 1:n){
    Y[i] <- (A*r[i]-B)/T
}

#Plot the results:

```

```

#Yield curves:
jpeg('CanadaPriceFit.jpg')
plot(Yield, type="l", ylim=c(min(Yield,Y),max(Yield,Y)),
main="Market yield vs th. yield, June 1st, 2006- Dec 18th, 2010",
xlab="Trading days from June 1st, 2006", ylab="ZCB yield", lwd=3)
points(Y, col="#9F9F9F", type="l", lwd=3)
legend(x="topright",col=c("black","#9F9F9F"), legend =
c("Market yield","Th. yield"), lwd=3)
dev.off()

#Theoretical interest rate:
jpeg('intrate28.jpg')
plot(r, type="l", main="Th. interest rate June 1st, 2006- Dec 18th, 2010",
xlab="Trading days from June 1st, 2006", ylab="Int. rate", lwd=3)
dev.off()

```

Simulation of regime-switched interest rate: (Fig 7.3)

```

n <- 1000 #No of. time points to consider
dt <- 1/n #Time step
t <- seq(0,1,dt) #Time points

set.seed(123) #Brownian increments
dW <- rnorm(n,0,1)*sqrt(dt)

#Parameter values:
r0 <- 0.05
k <- 1.5
theta1 <- 0.034
sigma <- 0.065
theta2 <- 0.15
beta <- 0.06

```

```

#Collect par.values in a vector:
p <- c(r0,k,theta1,sigma,theta2,beta)

r1 <- rep(0,n) #Regime-switched rate
r2 <- rep(0,n) #Original rate
W <- rep(0,n)
Ws <- rep(0,n)
dWs <- rep(0,n)

b <- function(x){
  if(x<p[6]){
    p[2]*(p[3]-x)
  }
  else{p[2]*(p[5]-x)}
}

r1[1] <- p[1] #Initial value of both
r2[1] <- p[1] #int.rate paths.

#Euler discretization of the interest rates:
for(i in 2:n){
  W[i] <- W[i-1]+dW[i-1]
  dWs[i] <- dW[i-1]+(1/p[4])*b(p[1]+p[4]*Ws[i-1])*dt
  Ws[i] <- Ws[i-1]+dWs[i-1]
  r1[i] <- p[1]+p[4]*Ws[i]
  r2[i] <- r2[i-1]+p[2]*(p[3]-r2[i-1])*dt+p[4]*dW[i]
}

#Plot of the two paths:
jpeg('RSvsNRS_IntRate.jpg')
plot(t[1:n], r1, type="l", main = "Regime-switched interest rate (black)
and non-regime switched interest rate (grey)",
xlab="Time", ylab = "Interest rates", ylim=c(min(r1,r2),max(r1,r2)),lwd=2)
lines(t[1:n], r2, col="#9F9F9F",lwd=2)
dev.off()

```

Pricing of default-free ZCB with(out) regime-switched interest rate (Fig. 7.4)

```

n <- 1000 #No of time points
dt <- 1/n #Time step
t <- seq(0,1,dt) #Time points

```

```

N <- 10000 #No. of MC repetitions
T <- 1 #Maturity of the ZCB.

#Model parameters:

r0 <- 0.05
k <- 1.5
th1 <- 0.034
th2 <- 0.15
s <- 0.065
beta <- 0.06

#b function for the
#regime-switched
#interest rate:

b <- function(x){
  ifelse(x<beta, k*(th1-x), k*(th2-x))
}

#b function for the
#non-regime-switched
#interest rate:

b2 <- function(x){
  k*(th1-x)
}

#Brownian increments
dW <- matrix(rnorm(n*N,0,1)*sqrt(dt), nrow=n)
W <- mat.or.vec(n,N)

#dWs* for the regime-switched
#interest rate
#corresponding to the function
#b:

dWs <- mat.or.vec(n,N)
Ws <- mat.or.vec(n,N)

```

```

#dWs* for the non-
#regime-switched interest rate
#corresponding to the function
#b2=k*(th1-x):

dWs2 <- mat.or.vec(n,N)
Ws2 <- mat.or.vec(n,N)

#Matrices for regime-switched (rs)
#and non-regime-switched (r) interest
#rates:

rs <- mat.or.vec(n,N)
r <- mat.or.vec(n,N)

#Initial values:
rs[1,] <- r0
r[1,] <- r0

#Euler discretization
#of both interest rates
#and the Brownian motions:
#(given by 7.2 and 7.3):

for(j in 1:N){
for(i in 2:n){
  #W_t under the measure Q:
  W[i,j] <- W[i-1,j]+dW[i-1,j]

  #P-Brownian motion Ws (from b2),(7.4):
  dWs[i,j] <- dW[i-1,j]+(1/s)*b(r0+s*Ws[i-1,j])*dt
  Ws[i,j] <- Ws[i-1,j]+dWs[i-1,j]

  #Non-RS interest rate (from 7.3):
  rs[i,j] <- r0+s*Ws[i,j]

  #P-Brownian motion, Ws2 (from b),(7.4):
  dWs2[i,j] <- dW[i-1,j]+(1/s)*b2(r0+s*Ws2[i-1,j])*dt
  Ws2[i,j] <- Ws2[i-1,j]+dWs2[i-1,j]

```

```

        #Non-RS interest rate (from 7.3):
        r[i,j] <- r0+s*Ws2[i,j]
    }}

#The three integrals in (7.5):
#For the RS interest rate:
I1 <- mat.or.vec(n,N)
I2 <- mat.or.vec(n,N)
I3 <- mat.or.vec(n,N)

#For the non-RS interest rate:
I1_R <- mat.or.vec(n,N)
I2_R <- mat.or.vec(n,N)
I3_R <- mat.or.vec(n,N)

#Matrices for prices:
#RS price:
Price_RS <- mat.or.vec(n,N)
#Non RS price:
Price_NRS <- mat.or.vec(n,N)

for(j in 1:N){
  for(i in 1:(n-1)){
    #Regime-switch:
    I1[i,j] <- -((T-i*dt)/(n-i+1))*sum(rs[i:n,j])
    I2[i,j] <- -(1/s)*sum(b(rs[i:n,j])*dWs[i:n,j])
    I3[i,j] <- ((T-i*dt)/(2*s**2))*(1/(n-i+1))*sum((b(rs[i:n,j]))**2)
    Price_RS[i,j] <- exp(I1[i,j]+I2[i,j]+I3[i,j])

    #Non-regime switch:
    I1_R[i,j] <- -((T-i*dt)/(n-i+1))*sum(r[i:n,j])
    I2_R[i,j] <- -(1/s)*sum(b2(r[i:n,j])*dWs2[i:n,j])
    I3_R[i,j] <- ((T-i*dt)/(2*s**2))*(1/(n-i+1))*sum((b2(r[i:n,j]))**2)
    Price_NRS[i,j] <- exp(I1_R[i,j]+I2_R[i,j]+I3_R[i,j])
  }}

#Calculating the mean, yielding
#the Monte Carlo prices:
P_RS <- rowMeans(Price_RS[1:(n-1),])
P_NRS <- rowMeans(Price_NRS[1:(n-1),])

```

#Plot the results (Fig. 7.2):

```
y1 <- min(P_RS, P_NRS)
y2 <- max(P_RS, P_NRS)

jpeg('RS_vs_NRS.jpg')
plot(t[1:(n-1)], P_NRS, type="l", xlab="Time", ylab="Price", main=
"Default-free prices, under RS and NRS interest rate",ylim = c(y1,y2),
lwd=3, col="#9F9F9F")
lines(P_RS, col="black", lwd=3)
legend(x="topleft", legend=c("N-RS ZCB price","RS ZCB price"), lty=1,
col = c("#9F9F9F","black",lwd=5))
dev.off()
```

Code for calibrating default-free ZCBs via regime-switch: (Fig. 7.5-7.6.)

```
T <- 0.25 #Maturity in yrs.
Price <- exp(-0.25*Yield) #Convert to price
n <- length(Price) #No. of time points
dt <- 1/n #Time step
tau <- 3 #No. of steps to integrate
y <- T/(tau+1)
r <- mat.or.vec(n,N) #Int. rate matrix
P <- mat.or.vec(n-tau,N) #Price matrix

#Gaussian increments under Q:
dW <- matrix(rnorm(n*N,0,1)*sqrt(dt), nrow=n)
W <- mat.or.vec(n,N)
Ws <- mat.or.vec(n,N)
dWs <- mat.or.vec(n,N)

#W_t under the measure Q:
for(j in 1:N){
  for(i in 2:n){
    W[i,j] <- W[i-1,j]+dW[i-1,j]
  }
}
```



```

#Function to minimize wrt. parameters:
f <- function(k){
  b <- function(x){
    ifelse(x<k[6], k[2]*(k[3]-x), k[2]*(k[5]-x))
  }
  r[1,] <- k[1] #Initial value of interest rate.
  for(j in 1:N){
    for(i in 2:n){
      #dWs* as given by (7.4)
      dWs[i,j] <- dW[i-1,j]+(1/k[4])*b(k[1]+k[4]*Ws[i-1,j])*dt
      Ws[i,j] <- Ws[i-1,j]+dWs[i-1,j]
      #The interest rate under the P (7.3):
      r[i,j] <- k[1]+k[4]*Ws[i,j]
    }
  }
  for(j in 1:N){
    for(i in 1:(n-tau)){
      #The price B(0,T) given by (7.5):
      P[i,j] <- exp(-y*sum(r[i:(i+tau),j]+
        0.5*(1/(k[4]**2))*(b(r[i:(i+tau),j]))**2))*
        exp(-(1/k[4])*sum(b(r[i:(i+tau),j])*dWs[i:(i+tau),j]))
    }
  }
  #The yield:
  Y <- -log(rowMeans(P))/T
  #The objective function:
  sum((Y-Yield[1:(n-tau)])**2/n)
}

#Run optimizer:
o <- optim(c(0.036576, 0.014484, 0.001343 , 0.072341 , 0.005617 , 0.048798),f)
#Save calibrated parameters:
p <- as.numeric(o$par)

#Repeating everything above in
#order to find the corresponding
#yield and interest rate paths,
#given the calibrated parameters:

r <- mat.or.vec(n,N)
P <- mat.or.vec(n-tau,N)
k <- p

```

```

b <- function(x){
  ifelse(x<k[6], k[2]*(k[3]-x), k[2]*(k[5]-x))
}

r[1,] <- k[1]

for(j in 1:N){
  for(i in 2:n){
    dWs[i,j] <- dW[i-1,j]+(1/k[4])*b(k[1]+k[4]*Ws[i-1,j])*dt
    Ws[i,j] <- Ws[i-1,j]+dWs[i-1,j]
    r[i,j] <- k[1]+k[4]*Ws[i,j]
  }}

for(j in 1:N){
  for(i in 1:(n-tau)){
    P[i,j] <- exp(-y*sum(r[i:(i+tau),j]+
      0.5*(1/(k[4]**2))*(b(r[i:(i+tau),j]))**2))*
      exp(-(1/k[4])*sum(b(r[i:(i+tau),j])*dWs[i:(i+tau),j]))
  }}

r2 <- rowMeans(r)
P2 <- rowMeans(P)
Y <- -log(P2)/T
Ysave <- -log(P2)/T #Calibrated yield
int <- r2 #Calibrated interest rate

#Plot the results:

jpeg('CanadaPriceFit2.jpg')
plot(Yield, type="l", ylim=c(min(Yield,Ysave),max(Yield,Ysave)),
  main="Market yield vs th. yield, June 1st, 2006- Dec 18th, 2010",
  xlab="Trading days from June 1st, 2006", ylab="ZCB yield", lwd=2)
points(Ysave, col="red", type="l", lwd=2)
legend(x=topleft, col=c("black","red"), legend = c("Market yield","Th. yield"), lwd=3)
dev.off()

jpeg('intrate282.jpg')
plot(int, type="l", main="Est. interest rate June 1st, 2006- Dec 18th, 2010",
  xlab="Trading days from June 1st, 2006", ylab="Int. rate", lwd=2)
dev.off()

```

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